Deposits in a Digital Era: Central Bank Digital Currency, Capital Structure, and Bank Risk Taking

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Abstract

There has been concern about how new digital money competing with bank deposits could impact the quality of bank lending. I develop a model to study how the usage of deposits as a means of payment affects the capital structure and effort of banks with risky investments. Because deposits are used in transactions and carry a liquidity premium, they are a relatively inexpensive source of funding for banks. The model endogenizes this liquidity premium and shows how the provision of deposits creates a moral hazard problem. When deposits are necessary for payments, banks will never exert the socially optimal level of effort. The effect of introducing a new payment asset that could raise banks' funding costs, such as a central bank digital currency (CBDC), depends on banks' net worth. If banks turn to capital funding as the cost of deposits increases, the riskiness of bank lending decreases. But if banks do not have enough net worth to raise more capital funding, riskiness can instead increase. This highlights the potential tradeoffs between improving payment efficiency and lowering the quality of bank credit for total welfare.

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1 Introduction

As the world becomes more digital and the share of payments made electronically continues to increase[1], central banks around the world have been considering issuing central bank digital currencies (CBDC)[2] A retail CBDC would be a central bank liability that is widely available to the public to use as payment in electronic transactions. However, digital payments currently rely mainly on commercial bank deposits, and a CBDC that could be a close substitute for bank deposits raises concern about the effect of introducing a CBDC on bank lending[3]. A competing CBDC in payments, especially if it pays an interest rate, could increase bank funding expenses by raising the cost of deposits or forcing banks to turn to sources of funding with higher costs, which can then impact lending. Existing studies have studied the extent to which banks may be disintermediated (i.e. how much they must reduce lending when they lose deposits to CBDC), but have not analyzed how an increase in funding costs will affect the risk taking of banks.

Bank lending is risky, and a large literature has studied factors that contribute to that risk. One argument for allowing banks to have profits or franchise value is that it will reduce risk-taking, as banks try to lower the risk of failure to not lose their profits. Therefore an increase in funding costs that lowers banks' profit margins could increase bank risk-taking. However, it is important to note that the one major source of funding banks can turn to is their own capital. Banks borrow money from households by issuing deposits, and if they do not need to repay depositors when they fail, there is an incentive to take on more risk. If banks are instead using their own capital as funding, they have a higher stake in the payoffs of bank loans and will want to reduce the risk of failure. If the introduction of a CBDC leads to an increase in capital funding, then it could also lead to lower bank risk-taking. However, banks may not be able to freely adjust their capital structure. It can be costly or slow to raise more equity, and the amount of capital they can invest may depend on their existing equity or net worth. In this paper, I study how the introduction of a new payment like CBDC will impact the riskiness of bank lending and how the banks' ability to change their capital structure matters for their decisions. To do so, we need to understand how changes in payments affecting the banks' liability side affects their behavior on the asset side.

I construct a model in which private bank deposits are essential for exchange and banks operate risky productive projects. This builds upon the framework of Lagos and Wright (2005)

[1]Although cash use has remained stable in the U.S., the Federal Reserve Diary of Consumer Payment choice finds increased credit card usage, remote payments, and payments made with mobile devices. Credit and debit cards combined accounted for about 65% of payments in 2024.

[2]According to the BIS in 2024, 91% of 93 surveyed central banks are exploring a CBDC.

[3]See, for example, the Federal Reserve's discussion paper of a potential CBDC, Money and Payments: The U.S. Dollar in the Age of Digital Transformation.

where money is necessary for exchange. Banks can issue deposits that serve as a medium of exchange, and can use deposits along with capital to fund productive projects that require some initial investment. The amount of capital funding banks can use will depend on their initial wealth (i.e. net worth or equity). The success of these projects will also depend directly on banks' choice of effort. When using deposits, banks have limited liability and do not repay depositors when projects fail. Because deposits can facilitate exchange, they can carry a liquidity premium which lowers the interest rate banks must pay on deposits and thus lowers the banks' cost of using deposits to fund investment. However this also gives banks an incentive to shift the risk of failure onto depositors, leading to a moral hazard problem in the banks' choice of effort.

First, I establish how a bank decides on its capital structure in an equilibrium without CBDC. When deposits are the only medium of exchange, banks will always issue deposits. When the total number of invesment opportunities in the economy is low, the total amount of deposits banks will be able to issue to fund these projects will also be low. The scarcity of liquid payment assets creates a large liquidity premium and lowers the interest rate on deposits in equilibrium. In this case, banks will choose to use lower amount or even zero capital financing. When the total number of investment opportunities is high though, and deposits are more abundant, the liquidity premium will be small and deposits will be more expensive. Banks will use a lower share of deposit funding and use a higher share of capital funding instead. Because of households' liquidity needs, banks will always issue deposits in equilibrium, and there will always be a risk-shifting incentive that makes bank lending risky. In addition, the amount of capital funding banks can use may be constrained by their wealth, which means they must rely on some deposit funding.

Then I consider how the introduction of an interest-bearing CBDC affects bank lending and household exchange. One main benefit of a CBDC is that it can be used by households to purchase goods. The central bank's choice of the interest rate it will pay on a CBDC will allow the central bank to influence the efficieny of exchange between households. If the CBDC is designed to pay a higher rate of return than bank deposits, then households will be able to pay for a larger amount of consumption goods. The effect of CBDC on bank lending will depend on how wealth- or equity-constrained banks' capital structure is.

When capital structure is not wealth-constrained, introducing a CBDC with a higher rate of return lowers the risk of projects failing. Deposit interest rates will adjust to match the CBDC interest rate in equilibrium and in this case, banks respond to the increase in deposit funding cost by switching to capital financing. By using more capital and thus having more skin in the game, banks exert more effort to avoid bank failure. But if the amount of capital

banks can use is limited by their wealth, then they may not be able to switch to more capital financing. In that case, the higher cost of deposit funding decreases banks profits, and banks have a lower incentive to prevent projects from failing and lower their effort. If banks are very wealth-constrained, relative to the total investment projects in the economy, and the cost of deposits is relatively high, they may be forced to reduce the total amount of lending.

Therefore, the effect of a CBDC on aggregate welfare will depend on how equity-constrained banks are. The central bank can improve payment efficiency and investment efficiency by increasing the interest rate on a CBDC when banks are not equity-constrained. In that case, the optimal policy decision would be to implement the Friedman rule. If banks are equity-constrained, the central bank must balance improvements in payment efficiency with reductions in investment efficiency when setting the rate paid on CBDC. If total investment opportunities in the economy are relatively small and the existing supply of liquid payment assets is also small, then the benefit to improving exchange between households by providing another medium of exchange may lead to an overall increase in welfare. However if payment assets are already relatively abundant, this benefit is not as large and the reduction in bank effort or lending will lead to an overall decrease in welfare. Then the optimal policy rate for CBDC when banks are wealth-constrained will be below the Friedman rule. While the focus here is on CBDC, which gives the central bank a policy tool via its interest rate, the analysis here can also be applied to understand how other digital payments such as stablecoins will impact the riskiness of the banking sector.

1.1 Related literature

This paper will contribute to the growing literature on CBDCs. My paper is most similar to those that also build on New Monetarist models to study the introduction of a CBDC. Keister and Sanches (2023) analyzes how CBDCs impact bank investment with a competitive banking sector, and Chiu et al. (2023) does so considering bank market power. While not New Monetarist, Andolfatto (2021) also studies this question for monopolist banks. Williamson (2022) and Monnet, Petursdottir, and Rojas-Breu (2025) also consider the bank risk dimension. Like in my model, Williamson (2022) considers incentive problems in banking and analyzes how CBDC affects safe collateral in the economy, but finds disintermediation can improve welfare by reducing overinvestment. Monnet, Petursdottir, and Rojas-Breu (2025) study a setting in which banks have a portfolio choice between safe and risky assets, and CBDC can instead cause overinvestment in safe assets. I add to this literature by considering the use of capital as a source of bank funding and how the banks' capital structure decisions impact bank risk through

their effort decisions.

There is also a large, well-established literature that recognizes moral hazard problems in bank lending. The seminal paper by Holmstrom and Tirole (1997) presents a model in which financial intermediaries can provide monitored lending to firms. The amount of monitored lending intermediaries provide is limited by the amount of capital they have. Intermediaries must invest enough of their own capital when monitoring to ensure incentives are aligned with borrowers. In this paper, banks will also hold capital to show depositors they have sufficient "skin in the game" for exerting effort in investments. Boyd and De Nicolo (2005) analyze how bank competition affects banks' risk incentives and note that the result depends on whether bank lending is treated as a portfolio problem or a contracting problem. In a portfolio problem, higher competition raises bank funding costs, lowers profits, and risk-seeking increases. But if banks also compete on loan rates and entrepreneurs are contracted to run projects, more competition lowers loan rates and thus entrepeneurs lower the risk of their investment projects. While I am not considering bank market power in this model, bank profits and the equity ownership of investment projects are also important for risk-taking and I am able to include elements of both problems and their opposing effects in the same model.

Both Martinez-Miera and Repullo (2017) and Dell'Ariccia, Laeven, and Marquez (2014) study how low interest rates can lead to increased risk-taking by banks. Martinez-Miera and Repullo (2017) consider how increased total saving in the economy lowers interest rates, leading banks to "search for yield" and increase risk-taking to reach higher returns. This paper is most closely related to Dell'Ariccia, Laeven, and Marquez (2014), which studies how changes in interest rates affect bank capital structure and risk-taking. A reference interest rate changes the cost of both deposit and equity, and lower rates lead to banks using more deposits and more risk-taking. I build on their model by endogenizing the liquidity premium that determines the cost of deposit funding. By explicitly modeling the payment aspect of deposits, I am able to consider how the introduction of another means of payment interacts with the banks' funding costs and risk choices, rather than through an exogenous reference rate.

Bank capital structure and capital requirements have been studied extensively in broader banking literature as well. In Allen, Carletti, and Marquez (2015), the authors consider the role of bank capital when there are bankruptcy costs. While they also endogenize the cost of deposit and equity financing for banks, they abstract from the liquidity of bank deposits. Gale and Gottardi (2020) also study how bank capital structure can be beneficial when there is costly default. Both Van den Heuvel (2008) and Begenau (2020) consider the costs and benefits of bank capital requirements. In Van den Heuvel (2008), the author considers the welfare costs of capital requirements. He similarly considers the household preferences for liquidity and how

equity capital reduces the banks' capacity to provide that liquidity, but focuses on the moral hazard probem from deposit insurance. Begenau (2020) provides quantitative evidence on bank capital requirements when banks also have risk-taking incentives. In her model, capital also improves monitoring incentives that lowers risk. However my paper does not explicitly focus on capital requirements and banks in my model may hold capital even without such requirements.

The rest of the paper is organized as follows. Section 2 presents the model and its main assumptions, and Section 3 characterizes the equilibrium. Section 4 defines and discusses the equilibrium without CBDC. In Section 5, I analyze the effects of introducing a CBDC on equilibrium allocations and how it impacts aggregate welfare. Section 6 concludes.

2 Environment

Time is discrete and continues forever. Each time period is split into two sub-periods, the CM and the DM. In the CM, agents interact in a Walrasian, centralized market, while in the DM, agents interact in a frictional, decentralized market. There is a perishable good in each period: the CM good x_t (also the numeraire) and the DM good q_t. The economy is populated by a continuum each of buyers, sellers, and banks that all discount the future with the same discount factor $\beta \in (0, 1)$. There is a consolidated fiscal/monetary government that I will also refer to as the central bank.

2.1 Buyers and sellers

Buyers and sellers (collectively, households) are infinitely-lived and participate in both the CM and DM. Both buyers and sellers can produce and consume in the CM, and the CM good can be produced one-for-one with a unit of labor h. In the DM, buyers want to consume but cannot produce, while sellers do not want to consume, but can produce. Buyers' preferences are

$$\sum_{t=0}^{\infty} \beta^t [x_t^b + u(q_t)].$$

The buyers' utility function for DM consumption is concave, with $u'(q) > 0$ and $u''(q) < 0$. We also assume that $-qu''(q)/u'(q) < 1$.

Sellers produce q at a linear, one-for-one cost. Their preferences are

$$\sum_{t=0}^{\infty} \beta^t [x_t^s - q_t].$$

In the DM, buyers are randomly matched pairwise with sellers and buyers make a take-it-or-leave-it offer to sellers to trade. For simplicity, I assume that a buyer will always match with a seller. I also assume buyers and sellers are anonymous in meetings, so there is a lack of commitment between households and no record-keeping and a medium of exchange is necessary. Sellers will accept both bank deposits and central bank digital currency (CBDC) as payment. During the CM, buyers can acquire liquid assets by producing the CM good and exchanging it for deposits with banks or for CBDC with the central bank. Sellers can use deposits or CBDC received as payment to purchase goods in the following CM.

2.2 Banks

Banks live for 1 period (or 3 sub-periods): they are born in the CM in period t and survive until the following CM in time $t+1$. Banks are identical and are each endowed with some wealth W at birth. I will refer to W as wealth throughout this paper, but it can also be interpreted as the banks' net worth or equity. Banks are the only agents who can operate productive risky projects and can choose to operate up to $I > 0$ total projects.

Each project requires a unit of initial investment, which can be funded by capital or debt. Let $k \in [0, 1]$ be the fraction of this initial investment funded by capital. I assume that the amount of capital a bank can raise is limited by their wealth. In order to fund ℓ loans using a capital ratio of k, they must have at least $W \geq k\ell$ in wealth[4]. Banks can consume any wealth that is not invested into the productive projects. Banks are not anonymous and they can credibly commit to repaying their debt if their projects succeed. They issue their debt in the form of bank deposits, which households can use as a medium of exchange. However, banks also have limited liability and do not repay depositors when their projects fail. Banks have a monopoly over productive projects and can earn positive profits, but they are perfectly competitive in the deposit market and take the price of deposits as given.

Projects are risky and have a chance of failing. Let $y > 0$ be the output of a successful project. I assume that banks can choose a level of effort s such that the project succeeds and produces y with probability s, and fails and produces 0 with probability $1 - s$. This effort is

[4]This can be interpreted as the bank facing an infinite cost to raise capital beyond W.

costly, and I assume that the cost of effort is convex and satisfies the following assumption.

Assumption 1. *For a level of effort s, the cost of effort $c(s)$ is such that $c(0) = 0$, $c'(s) > 0$, $c''(s) > 0$, and $c'''(s) \geq 0$.*

This cost of effort can be interpreted as a cost of monitoring projects to ensure that they succeed. I also make the following assumption about the project output:

Assumption 2. *Project payoffs y are such that*

$$\beta \max_s \{sy - c(s)\} > 1$$

This assumption states that there exists a level of effort s which satisfies $y - c'(s) = 0$ at which the bank can make positive profits and operate.

Households will exchange δ units of CM goods to the banks for δ units of deposits. Banks promise to repay depositors a total of Z on δ units of deposits in the following CM, so the listed deposit rate is $R_d = Z/\delta$. Households are aware that the banks' projects may fail, but they cannot directly contract on or observe s. They can infer the bank's effort by observing their capital ratio k and expect that they will repaid with probability $\mathbb{E}[s|k]$. Therefore, the implied deposit rate with risk is R, where $R\delta = \mathbb{E}[s|k]Z$.

The timeline of bank operations is as follows: in their first CM when young, banks decide on how many projects to operate and on the capital structure of their funding. From there, they issue deposits to households and raise capital before the DM. At the end of the DM, after buyers and sellers have made their exchange, banks make their effort choice s. The outcome of the project realizes at the beginning of the following CM, when banks are old. If projects are successful, depositors are repaid and banks consume their profit. If projects fail, then depositors receive nothing and banks do not consume. Note that if banks fail and deposits are not repaid, there is no effect on DM exchange. Preferences are linear in the CM and households have already anticipated the risk of failure in the implied deposit rate R[5]. Old banks then die and households obtain deposits from the next generation of banks before the next DM.

[5] An alternate explanation is that R can be thought of as a competitive deposit rate that must be offered by all banks and depositors can split their deposits across an arbitrarily large number of banks. Across all of their deposit holdings, they receive sZ in total repayment and lose $(1 - s)Z$.

2.3 Government

The government can issue CBDC, which is currency available to all agents and pays an interest rate of R_e. The government also has the ability to tax households, which is used to pay for the interest on CBDC. The aggregate supply of the central bank digital currency E is set such that

$$E' = R_e E + 2\tau \tag{1}$$

where the τ is the lump-sum tax on both buyers and sellers and prime denotes variables in the next period.

2.4 Welfare

Aggregate welfare per period is defined as:

$$\mathcal{W} = \{-1 + \beta[sy - c(s)]\}\ell + x^b + x^s + \beta[u(q) - q]. \tag{2}$$

The households' quasi-linear preferences imply that the distribution of CM consumption will not impact welfare. For discussions of optimal policy, we establish that the optimal allocation of q, s, and ℓ is the solution to an unconstrained social planner's problem:

$$\max_{q,s,\ell} \{-1 + \beta[sy - c(s)]\}\ell + \beta[u(q) - q]$$

subject to the feasibility constraint $\ell \leq I$.

To maximize the trade surplus between buyers and sellers, the planner would set $q = q^*$, where $u'(q^*) = 1$. To maximize the output from bank projects, the planner runs all projects available $\ell = I$ and sets bank effort such that $s = s^*$, where $y - c'(s^*) = 0$.

3 Equilibrium analysis

3.1 Buyers' problem

Let $\delta \in \mathbb{R}_+$ be a buyer's holding of bank deposits and let $e \in \mathbb{R}_+$ be their holding of CBDC. In the beginning of the CM, a buyer with a portfolio of (δ, e) has the value function W^b

$$W_t^b(\delta, e) = \max_{\delta', e', x^b, h} \{x^b - h + \beta V_{t+1}^b(\delta', e')\} \tag{3}$$

subject to their budget constraint $x + \delta' + e' + \tau = h + R\delta + R_e e$.

The buyer's CM consumption x, taxes τ, and payment balances for the next DM δ' and e' are financed by labor and previous payment balances. By substituting $x - h$ from the budget constraint into the value function, W^b simplifies to

$$W_t^b(\delta, e) = R\delta + R_e e + \tau + \max_{\delta', e'} \{-\delta' - e' + \beta V_{t+1}^b(\delta', e')\}. \tag{4}$$

At the beginning of the DM, the buyer's value function is given by

$$V_t^b(\delta, e) = u[q(\delta, e)] - p(\delta, e) + W_t^b(\delta, e) \tag{5}$$

where $q(\delta, e)$ is the amount of DM goods the buyer can afford with their portfolio of liquid assets, and $p(\delta, e)$ is the amount of liquid assets given to sellers as payment.

In the DM, buyers make take-it-or-leave-it offers to sellers. The terms of trade are given by

$$\max_{q,p} \{u(q) - p\}$$

subject to the seller's participation constraint $p \geq q$ and the buyer's liquidity constraint$p \leq R\delta + R_e e$.

The solution is:

$$q(\delta, e) = \begin{cases} q^* & \text{if } R\delta + R_e e \geq q^* \\ R\delta + R_e e & \text{otherwise} \end{cases}.$$

where q^* satisfies $u'(q^*) = 1$. If the value of the buyer's liquid assets is large enough to pay sellers to produce q^*, the efficient level of trade occurs. If not, the buyer spends all of their liquid assets and the seller will produce $q < q^*$.

Substituting (5) into (4) and using the linearity of W^b, the buyer's CM value function becomes

$$W_t^b(\delta, e) = R\delta + R_e e - \tau + \beta W_t^b(0,0) + \max_{\delta', e'}\{-\delta' - e' + \beta[u(q(\delta', e')) - p(\delta', e') + R\delta' + R_e e']\}$$

The first four terms are independent of δ' and e', so the buyer's portfolio problem can be written as:

$$\max_{\delta, e}\{-\delta - e + \beta[u(q(\delta, e)) - p(\delta, e) + R\delta + R_e e]\}$$

subject to $q \leq R\delta + R_e e$, $\delta \geq 0$, and $e \geq 0$.

If the buyer's assets are sufficient to purchase q^*, this implies that either $R = \frac{1}{\beta}$ or $R_e = \frac{1}{\beta}$. Buyers will only hold enough liquid balances for q^* when it is costless to hold payment balances over time.

The first-order conditions for deposits can be written as

$$u'(q) \leq \frac{1}{\beta R} \tag{6}$$

with equality if $\delta > 0$. The first order condition for CBDC can be written as

$$u'(q) \leq \frac{1}{\beta R_e} \tag{7}$$

with equality if $e > 0$. These first-order conditions characterize the buyers' demand for liquid balances in the CM. Because sellers can accept both deposits and CBDC as perfect substitutes, buyers will choose to hold whichever payment has the highest return. If deposits and CBDC pay the same rate of return, then buyers are indifferent about which asset they hold.

3.2 Bank's problem

Since households must be correct about $\mathbb{E}[s|k]$ in equilibrium, I will use s in notation for the bank's effort or probability of success for the following analysis. Bank profits are given by

$$-k\ell + \beta[s(y\ell - Z) - c(s)\ell].$$

The first term captures the cost of initial capital investment for ℓ projects. The second term captures the payoff in the next period, net of effort cost and deposit repayments. The bank will choose how many projects or loans to undertake ℓ, their capitalization ratio k, their effort level s, and how much to repay depositors Z to maximize profits. If banks fund a fraction k of loans

with capital, then the other fraction $(1-k)$ is funded by deposits, and the total deposits the bank issues is $\delta^I = (1-k)\ell$.[6]

Suppose buyers could contract directly on effort and require that $sZ \geq R(1-k)\ell$. This constraint will always hold with equality (details can be found in the appendix) and therefore the problem of a bank maximizing profits can be written as

$$\max_{s,k,\ell} -k\ell + \beta[sy - R(1-k) - c(s)]\ell$$

subject to $W \geq k\ell$, $\ell \leq I$, and $0 \leq k \leq 1$.

The first-order condition for s is $\beta[y - c'(s)]\ell = 0$, which implies that the bank would choose the socially efficient level of effort, s^*. But because buyers cannot control the bank's effort, they must instead contract on a repayment amount Z that is incentive compatible for banks. The bank chooses a level of effort such that

$$s \in \arg\max_{s'} \{-k\ell + \beta[s(y\ell - Z) - c(s)\ell]\}.$$

An interior solution to this constraint can be characterized by the first-order condition

$$y\ell - Z - c'(s)\ell \geq 0.$$

When operating ℓ projects, the marginal payoff to another unit of effort $(y\ell - Z)$ must be at least as great as the marginal cost of another unit of effort $c'(s)\ell$. Then we consider the following bank problem:

$$\max_{s,k,\ell,Z} -k\ell + \beta[s(y\ell - Z) - c(s)\ell]$$

subject to the bank's incentive constraint,

$$y\ell - Z - c'(s)\ell \geq 0 \tag{8}$$

the buyers' participation constraint,

$$sZ \geq R(1-k)\ell \tag{9}$$

the wealth constraint,

$$W \geq k\ell \tag{10}$$

[6]This can alternatively be set up such that the bank invests a total of $\kappa = k\ell$ capital and issues a total of $\delta^I = (1-k)\ell$ to fund ℓ loans, so $\kappa + \delta = \ell$. This does not alter the results of the bank's problem.

and feasibility constraints on loans and the capitalization ratio:

$$0 \leq \ell \leq I$$
$$0 \leq k \leq 1.$$

Buyers will not hold deposits unless the expected repayment satisfies $sZ \geq R(1-k)\ell$, which I call the participation constraint. This constraint will always bind with equality, since the bank will not exert more costly effort than needed to attract depositors at R. Then we can combine (8) and (9) to eliminate Z from the bank's problem. Finally, the bank problem we consider is

$$\max_{s,k,\ell} -k\ell + \beta[sy - R(1-k) - c(s)]\ell \tag{11}$$

subject to

$$R(1-k)\ell \leq s(y - c'(s))\ell \tag{12}$$
$$W \geq k\ell \tag{13}$$
$$0 \leq \ell \leq I \tag{14}$$
$$0 \leq k \leq 1. \tag{15}$$

where the bank's incentive constraint (12) requires that expected marginal payoff for an extra unit of effort is at least as great as the marginal cost of an extra unit of deposit funding.

Let $\omega = W/I$ be the wealth to investment ratio. As we will see, the bank's solution will depend on their wealth relative to the total investment opportunities in the economy. Also let $J(s) \equiv s(y - c'(s))$ denote the bank's expected marginal payoff for effort so that the bank's incentive constraint can be written as $J(s) = R(1-k)$. Because banks are perfectly competitive in the deposit market, they take R as given. The bank's choice of s, k, and ℓ for different risk-adjusted deposit rates can be characterized in the following lemma.

Lemma 1. *For a given ω, there exists cutoffs $\underline{R}$, $\hat{R}(\omega)$, and $\bar{R}(\omega)$ such that the solution to the bank's problem can be characterized as follows:*

1. *(Region 0): If $R < \underline{R}$, the bank uses no capital ($k = 0$) and all wealth is consumed, all projects operate ($\ell = I$), and the capital structure is onconstrained by wealth. Effort s satisfies*
$$R = J(s) \tag{16}$$

2. *(Region 1): If $\underline{R} \leq R < \hat{R}(\omega)$, some wealth is consumed and k is interior, all projects*

operate ($\ell = I$), and capital structure is unconstrained. Effort s solves

$$\frac{1}{\beta R} - 1 = H(s), \tag{17}$$

and k solves

$$R(1-k) = J(s) \tag{18}$$

3. *(Region 2): If $\hat{R}(\omega) < R \leq \min\{\bar{R}(\omega), \frac{1}{\beta}\}$, all wealth is invested, all projects operate ($\ell = I$), and capital structure is constrained ($k = \omega$). Effort s satisfies*

$$R(1-\omega) = J(s) \tag{19}$$

4. *(Region 3): If*

$$\max_s J(s) < \frac{1}{\beta}$$

and $\omega < \omega_0$, then if $\bar{R}(\omega) < R < \frac{1}{\beta}$, all wealth is invested and some projects operate. The solution is given by a level of effort s that satisfies

$$sy - c(s) - R + H(s)[J(s) - R] = 0, \tag{20}$$

a capital ratio k such that

$$R(1-k) = J(s), \tag{21}$$

and loans satisfy

$$\ell = \frac{\omega I}{k} < I. \tag{22}$$

The boundaries are ordered such that $\underline{R} \leq \hat{R}(\omega) < \bar{R}(\omega)$, with the first inequality strict when $\omega > 0$. Both $\hat{R}(\omega)$ and $\bar{R}(\omega)$ are increasing with ω. If $\omega \geq 1$, the bank's solution will only fall in Regions 0 and 1. If $\omega < 1$, then the bank's solution can fall in all four regions.

The proof for the lemma is provided in the appendix. The bank's incentive constraint always binds. If $J(s) > R(1-k)$, the expected payoff for another unit of effort is greater than the cost, so the bank can exert another unit of effort and still increase profits. Therefore, the bank will increase effort until $J(s) = R(1-k)$ and the profits from deposit funding are maximized.

The bank's choice of capitalization ratio will be influenced by the marginal cost and benefit of using additional deposit funding. Let

$$H(s) = \frac{y - c'(s)}{sc''(s) - c'(s) - y}$$

denote this marginal cost of deposit funding. By issuing more deposits, banks shift the risk of failure onto depositors and exert less effort, which lowers the total project output. The marginal benefit to using deposits is

$$\frac{1}{\beta R} - 1 \tag{23}$$

Because capital must be invested upfront before the project payoffs realize in the next period, the opportunity cost of using a unit of wealth for capital instead of consuming it during the bank's first CM is $\frac{1}{\beta}$. When $R < \frac{1}{\beta}$, the cost of using deposit funding is lower than the cost of capital funding. Therefore banks have an incentive to use more deposits the lower R is.

Banks will also choose to run all available projects as long as bank profits are positive. This is the case for Regions 0-2. In Region 3, the bank makes zero profits and ℓ may be less than I. I discuss the details of each section below.

- *Region 0.* In this region, the deposit interest rate is low enough that the marginal benefit of using deposits is greater than the marginal cost

  $$\frac{1}{\beta R} - 1 > H(s)$$

  so banks are fully deposit financed. Because they do not use any capital financing, they do not face a wealth constraint. Effort s is chosen to satisfy their incentive constraint, $J(s) = R$. The marginal benefit to an additional unit of effort decreases as s increases, $J'(s) < 0$, and effort in this region decreases as R increases:

  $$\frac{ds}{dR} = \frac{1}{J'(s)} < 0.$$

  Since banks have invested no capital and thus have zero skin-in-the-game, they are fully exposed to a decrease in profits when the cost of deposits increase. Since profits when successful decrease, bank's level of effort also decreases.

- *Region 1.* In this region, banks decide on s such that the marginal benefit of using deposit from the lower funding cost is equal to the marginal cost of lowering output because of risk shifting:

  $$\frac{1}{\beta R} - 1 = H(s). \tag{24}$$

  From this level of effort, they choose a capital ratio to satisfy $R(1-k) = J(s)$. If banks issue deposits, they must repay depositors at a higher R, and they will need to invest some capital in order to have the incentive to do so. When some of the initial investment

comes from their own wealth, they will claim a larger share of the profits when successful (because they repay less deposits) and therefore have incentive to exert more effort. In this region, the amount of wealth banks have is larger than than the capital needed for investment.

As R increases in this region, effort increases

$$\frac{ds}{dR} = -\frac{1}{\beta R^2 H'(s)} > 0.$$

and k increases

$$\frac{dk}{dR} = -\frac{R}{1-k}\left[J'(s)\frac{ds}{dR}\right] > 0.$$

Note that if $\omega \geq 1$, the bank's total wealth is always at least as large as total investment opportunities $W \geq I$. Therefore, the bank will always have enough wealth to raise more capital, even up to being fully capital financed ($k = 1$). Then the following two regions will only exist when wealth is relatively scarce, $\omega < 1$ (or $W < I$.)

- *Region 2.* In this region, the marginal cost of deposits is greater than the benefit,

$$\frac{1}{\beta R} - 1 < H(s)$$

but banks are now wealth constrained and cannot use a higher capital ratio than $k = \omega$. If banks were not wealth-constrained, they would choose a higher capital ratio to balance the funding benefit of deposits against the moral hazard cost ($k > \omega$). However, because they are wealth-constrained, they can only invest a fraction ω of loans in capital and must issue a fraction $(1 - \omega)$ of loans in deposits. Effort is determined by the bank's incentive constraint $R(1 - \omega) = J(s)$ and like in Region 0, effort decreases as R increases

$$\frac{ds}{dR} = \frac{1-\omega}{J'(s)} < 0.$$

because the capital structure is constrained, and the increase in R only lowers the bank's profits without raising their stake in the profits if successful.

Bank profits will decrease in this region as R increases and s decreases. For interest rates in this region, the banks will still make a positive profit on each project, and will therefore operate all projects. If wealth is scarce enough, the level of R where the bank reaches zero profits $\bar{R}(\omega)$ will be less than $\frac{1}{\beta}$. Let ω_0 denote the wealth-investment ratio such that this is the case when $\omega < \omega_0$.

- *Region 3.* In this region, R is high enough that banks make zero profits on each loan

$$sy - c(s) - R + H(s)[J(s) - R] = 0.$$

where the left-hand side of the equation is the bank's profits when wealth constrained. Because the bank will not operate with negative profits, a higher level of R requires that banks increase effort to maintain zero profits:

$$\frac{ds}{dR} = \frac{1 + H(s)}{H'(s)[J(s) - R]} > 0.$$

Then given R and the required s for zero profits, the bank chooses a capital ratio to satisfy their incentive constraint $R(1-k) = J(s)$. Banks must exert more effort to secure deposit funding when $R > \bar{R}(\omega)$, and that requires a higher ratio of capital investment for banks to have the incentive to do so. This k is increasing in R in this region

$$\frac{dk}{dR} = -\frac{1}{R}\left[J'(s)\frac{ds}{dR} - (1-k)\right] > 0.$$

The bank is still wealth-constrained in this region, but banks do not have enough wealth to operate all I projects with the capital ratio they need to operate. Banks will therefore reduce the total number of projects they operate. Given the s and k needed to operate, the banks have enough wealth to run

$$\ell = \frac{\omega I}{k} = \frac{W}{k}$$

projects. As R increases, raising s and k, the number of projects banks can afford to fund for a given amount of wealth decreases.

The lower boundary of Region 3 may be greater than $R = \frac{1}{\beta}$. Let $\bar{R}_0 = \max_s J(s)$. If $\bar{R}_0 \geq \frac{1}{\beta}$, then the project payoff y is large enough that the bank has enough incentive to operate all projects without capital funding for any $R \leq \frac{1}{\beta}$. Even when banks must repay all of their initial investment at rate R, the projects will still be profitable. Therefore, for the boundary of Region 3 to be less than $R = \frac{1}{\beta}$, we must assume that $\bar{R}_0 < \frac{1}{\beta}$.

Let ω_0 be the level of wealth such that

$$\frac{1}{\beta}(1 - \omega_0) = J(s_0) \tag{25}$$

where s_0 is defined by

$$s_0 y - c(s_0) - \frac{1}{\beta} + H(s_0)\left[J(s_0) - \frac{1}{\beta}\right] = 0. \tag{26}$$

For wealth scarce enough, $\omega < \omega_0$, there is a region where $R < \frac{1}{\beta}$ such that banks operate with zero profits. For the remaining analysis in this section, I will assume that y is such that $\bar{R}_0 < \frac{1}{\beta}$. The boundaries between regions are depicted in Figure 1.[7]

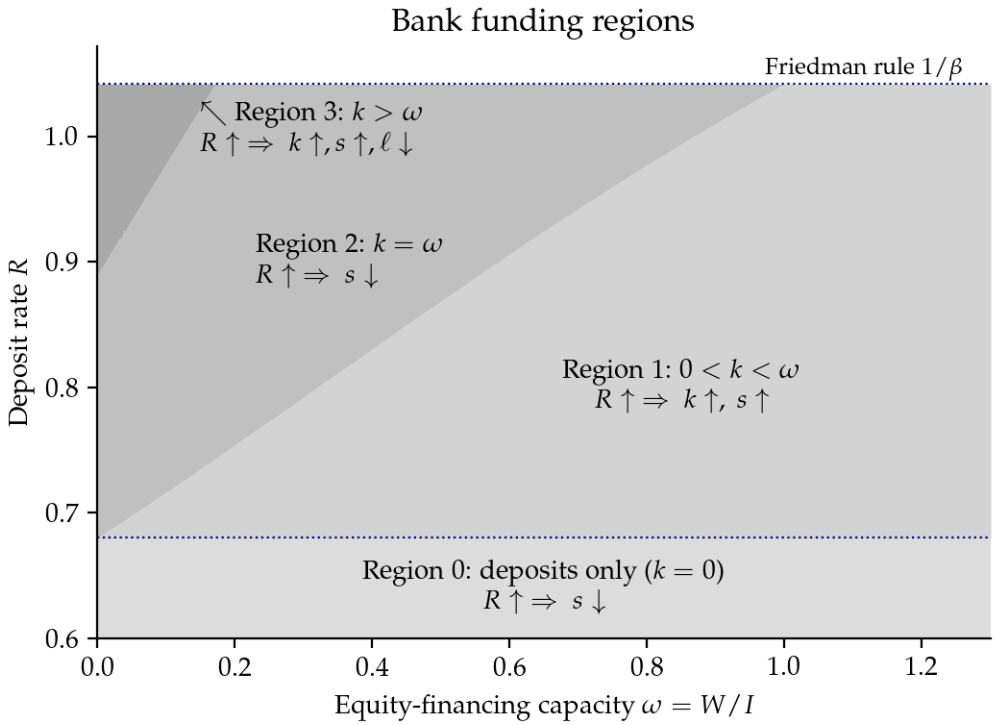
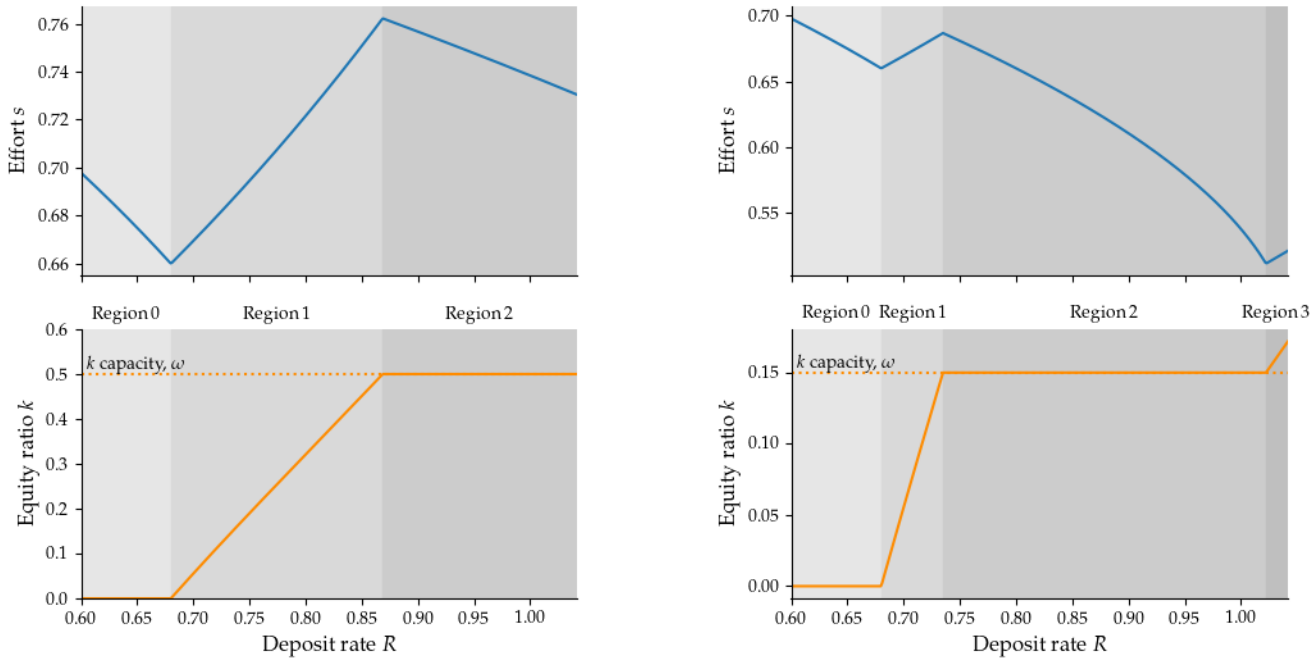


Figure 1: Bank solution regions as a function of ω.

The bank uses no capital when $R < \underline{R}$ and will not face a wealth constraint, so the upper boundary of Region 0 doesn't depend on ω. The boundaries between Regions 1 and 2 and between Regions 2 and 3 are increasing with ω. As wealth increases relative to investment, the bank will be wealth constrained only for higher deposit rates. And when $\omega \geq 1$, the bank will not be wealth constrained for any $R \leq \frac{1}{\beta}$ and so the bank solutions will only fall in Regions 0 or 1.

[7]The figure uses $u(q) = q^{1-\eta}/(1-\eta), c(s) = \gamma s^2/2, \eta = 0.5, \beta = 0.96, \gamma = 4.5$, and $y = 4$.



(a) $\omega = 0.5$ (b) $\omega = 0.15$

Figure 2: Capitalization and effort decisions, given R.

To summarize, the banks' capitalization ratio k is given by

$$k(R,\omega) = \begin{cases} 0 & \text{if } R \leq \underline{R} \\ \text{solution to equation (18)} & \text{if } \underline{R}(\omega) < R < \hat{R}(\omega) \\ \omega & \text{if } \hat{R}(\omega) \leq R < \bar{R}(\omega) \\ \text{solution to equation (21)} & \text{if } \bar{R}(\omega) \leq R < \frac{1}{\beta}. \end{cases} \tag{27}$$

The bank's choice of effort $s(R,\omega)$ will then solve

$$R(1 - k(R,\omega)) = J(s). \tag{28}$$

Figure 2 shows how the banks' capitalization decision and effort decision changes with R, taking ω as given.

The total number of projects the banks run ℓ is given by

$$\ell = \begin{cases} I & \text{if } R \leq \bar{R}(\omega) \\ \text{solution to equation (22)} & \text{if } \bar{(}R)(\omega) < R \end{cases} \tag{29}$$

Then for a given ω and R, the total amount of deposits banks will issue is $\delta^I = (1 - k(R,\omega))\ell(R,\omega)$.

4 Equilibrium without CBDC

First, suppose that the central bank is not issuing CBDC and bank deposits are the only medium of exchange. The buyers' demand for deposits is given by

$$\frac{1}{\beta R} = u'(q) \tag{30}$$

or alternatively, by the demand schedule

$$\delta = \begin{cases} \infty & \text{if } R > \frac{1}{\beta} \\ \beta q^* & \text{if } R = \frac{1}{\beta} \\ \frac{1}{R}(u')^{-1}\left(\frac{1}{\beta R}\right) & \text{if } R < \frac{1}{\beta}. \end{cases} \tag{31}$$

If $R > \frac{1}{\beta}$, there is infinite demand for deposits. If $R = \frac{1}{\beta}$, it is costless for buyers to hold deposits, so they hold enough deposits to afford q^*. If $R < \frac{1}{\beta}$, it is costly for buyers to hold deposits over time. They only hold enough deposits to satisfy (30) and will only be able to afford $q < q^*$.

For the market for deposits to clear, the amount of deposits demanded by households must equal the amount of deposits supplied by the bank, $\delta = \delta^I$, or

$$\delta = [1 - k(R,\omega)]\ell(R,\omega) \tag{32}$$

Definition 1. *An equilibrium without CBDC consists of the price R, quantities q, s, ℓ and the ratio k that solve* (27), (28), (29), (30), *and market clearing* (32).

Definition 1 gives the conditions necessary for equilibrium. To guarantee the existence of

equilibrium when the bank has no wealth and $\bar{R}_0 < \frac{1}{\beta}$, I make the following assumption:

$$(u')^{-1}\left(\frac{1}{\beta \bar{R}_0}\right) \geq \bar{R}_0 I$$

This assumption guarantees that there will be enough demand for deposits to meet the supply at some interest rate less than or equal to $\bar{R}_0$. The following following proposition establishes the existence and uniqueness of equilibrium in the economy without CBDC.

Proposition 1. *There exists a unique equilibrium in the economy without CBDC. Banks' level of effort s is always less than s^*. There is a liquidity premium on deposits in the following cases:*

(i) $\omega = 0$ *and* $\bar{R}_0 < \frac{1}{\beta}$*;*

(ii) $\omega = 0$ *and* $\bar{R}_0 \geq \frac{1}{\beta}$ *and* $I < \beta q^*$*;*

(iii) $\omega \geq 1$*;*

(iv) $\omega_0 < \omega < 1$ *and* $(1-\omega)I < \beta q^*$*; or*

(v) $0 < \omega \leq \omega_0$ *and* $(1-\omega)\frac{\omega}{\omega_0} I < \beta q^*$*;*

Proofs of all propositions can be found in the appendix. For the deposits market to clear, the bank always issues some deposits. However, this means that the bank's level of effort in equilibrium will always be less than the efficient level s^*. If banks are fully capital financed ($k = 1$) and receive all of the profits when they run their productive projects, $R(1-k) = 0 = s(y - c'(s))$ implies that they will have incentive to choose s^* to maximize output such that $y - c'(s) = 0$. But since issuing deposits requires $k < 1$, banks will always shift some risk to depositors and s will be less than s^*. This problem cannot be eliminated as long as banks are the only agens issuing liquid assets.

The equilibrium deposit interest rate will depend on the total productive projects I in the economy and how wealth-constrained the bank is. When the bank is not wealth-constrained ($\omega \geq 1$), there will always be a liquidity premium on deposits and $R < \frac{1}{\beta}$. Households will only hold enough deposits for q^* when $R = \frac{1}{\beta}$. Because the bank will always have enough wealth to fund all I projects through capital and do not have to rely on any amount of deposit funding, they do not provide any deposits when $R = \frac{1}{\beta}$. Doing so requires an even higher market deposit rate for banks. As a reminder, the bank must repay some amount Z such that $sZ = R(1-k)\ell$:

$$\frac{sZ}{(1-k)\ell} = sR_d.$$

If $R = \frac{1}{\beta}$, this means that the listed market rate for banks is even larger, $R_d = \frac{1}{s\beta}$, and banks will switch to capital financing instead. Therefore, for the banks to issue deposits, there must be a liquidity premium on deposits: $R^N < \frac{1}{\beta}$.

When banks have some wealth but not enough to fund all I projects with capital, then it may be possible for buyers to afford q^* if I is high enough. When banks are wealth-constrained, then they are forced to rely on some amount of deposit funding and cannot be completely capital-financed. For a given $\omega \in (\omega_0, 1)$, if I is large and the amount of deposits they must issue $(R(1-\omega)I)$ at $R = \frac{1}{\beta}$ is also sufficiently large, then there is no liquidity premium on deposits in equilibrium: $R^N = \frac{1}{\beta}$. But if I is not as abundant or if the bank's wealth relative to investment ω is larger and they are less reliant on deposits, such that $(1-\omega)I < \beta q^*$, then there will be a liquidity premium on deposits. If $\omega \in (0, \omega_0)$, banks are even more equity-constrained and issue less than I loans. At $R = \frac{1}{\beta}$, banks must exert effort $s = s_0$, as defined in (26) and choose $k = \omega_0$. Banks can only operate a fraction $\omega/\omega_0 < 1$ of total I. But if I is sufficiently high for a given ω, it is possible for banks to supply enough deposits for buyers to afford q^*. Otherwise, deposits are relatively scarce and there will be a liquidity premium.

If banks have no wealth and all investment must be funded by deposits, the liquidity premium of deposits depends on whether banks will operate when $R = \frac{1}{\beta}$. If they do (which requires $\bar{R}_0 \geq \frac{1}{\beta}$), then there is a liquidity premium only if total investments are scarce, $I < \beta q^*$. But if $\bar{R}_0 < \frac{1}{\beta}$, banks will not operate. They will only operate for $R < \frac{1}{\beta}$ and therefore, buyers will not hold enough deposits to afford q^*.

Proposition 1 establishes that an equilibrium without CBDC exists, and that either banks are wealth-constrained or buyers are liquidity-constrained. We can further distinguish four equilibrium regions that can occur, depending on I and ω.

Proposition 2. *Suppose that $\bar{R}_0 > \frac{1}{\beta}$ and wealth is sufficiently large relative to investment, $\omega \geq 1$. There exist a cutoff $\underline{I}$ such that equilibrium falls into one of two cases:*

1. *(Equilibrium region 0): If $I \leq \underline{I}$, then equilibrium is given by q, s, k, ℓ, and R such that*

$$q = RI \tag{33}$$

$$R = J(s) \tag{34}$$

$$k = 0 \tag{35}$$

$$\ell = I \tag{36}$$

$$1 = \beta R u'(RI). \tag{37}$$

2. *(Equilibrium region 1): If $I > \underline{I}$, then equilibrium is given by q, s, k, ℓ, and R such that*

$$q = R(1-k)\ell \tag{38}$$
$$\frac{1}{\beta R} - 1 = H(s) \tag{39}$$
$$R(1-k) = J(s) \tag{40}$$
$$\ell = I \tag{41}$$
$$1 = \beta R u'(R(1-k)I). \tag{42}$$

Proposition 3. *Suppose that $\bar{R}_0 < \frac{1}{\beta}$ and bank wealth is limited relative to total investment, $\omega < 1$. Then there exist cutoffs $\underline{I}$, $\hat{I}(\omega)$, and $\bar{I}(\omega)$ such that equilibrium falls into one of the following cases:*

1. *(Equilibrium region 0): If $I \leq \underline{I}$, then equilibrium is given by q, s, k, ℓ, and R such that* (33)-(37) *are satisfied.*

2. *(Equilibrium region 1): If $\underline{I} < I \leq \hat{I}(\omega)$, equilibrium is given by q, s, k, ℓ, and R such that* (38)-(42) *are satisfied.*

3. *(Equilibrium region 2): If $\hat{I}(\omega) < I \leq \bar{I}(\omega)$, equilibrium is given by q, s, k, ℓ, and R such that*

$$q = R(1-\omega)I \tag{43}$$
$$R(1-\omega) = J(s) \tag{44}$$
$$k = \omega \tag{45}$$
$$\ell = I \tag{46}$$
$$1 = \beta R u'(R(1-\omega)I). \tag{47}$$

4. *(Equilibrium region 3): If $\bar{I}(\omega) < I$, equilibrium is given by q, s, k, ℓ, and R such that*

$$q = R(1-k)\ell \tag{48}$$
$$sy - c(s) - R + H(s)[J(s) - R] = 0 \tag{49}$$
$$R(1-k) = J(s) \tag{50}$$
$$\ell = \frac{\omega I}{k} \tag{51}$$
$$1 = \beta R u'(R(1-k)\ell). \tag{52}$$

The boundaries are defined such that $\underline{I} < \hat{I}(\omega) < \bar{I}(\omega)$ *and* $\hat{I}(\omega)$ *and* $\bar{I}(\omega)$ *are increasing with* ω*.*

Proposition 2 defines equilibrium regions when bank wealth is relatively large, while Proposition 3 defines equilibrium regions when bank wealth is relatively small. These equilibrium regions can be distinguished based on the same interest rate cutoffs $\underline{R}$, $\hat{R}(\omega)$, and $\bar{R}(\omega)$ from the banks' problem, for a given wealth to total investment ratio. Total investments opportunities I matter because that determines the total amount of liquid deposits banks are able to supply. When I is larger, the total amount of deposits banks can issue will also be larger and the market for deposits will clear at a higher interest rate. I further discuss this by region below.

- *Equilibrium region 0.* In equilibrium region 0, the interest rate is below $\underline{R}$. This occurs when $I < \underline{I}$ and total investment opportunities are the most scarce. Therefore, the banks' supply of deposits is small relative to buyers' demands. The scarcity of deposits issued by the bank leads to a large liquidity premium, and the deposits market clears at a low interest rate. Buyers are willing to hold deposits even if they offer a low rate of return, because they would like to purchase the DM good q.However, they will not carry enough deposits to afford q^* in the DM. On the banks' side, banks choose to be fully deposit financed because the funding cost benefit to using deposits at a low R over capital is greater than the moral hazard cost on their total output. Effort s follows from the bank's solution in region 0 and they will operate all projects. Note that because banks do not use any capital in this case, this cutoff for investment $\underline{I}$ does not depend on the banks' wealth and is the same for any ω.

- *Equilibrium region 1.* In this region, the interest rate falls between $\underline{R}$ and $\hat{R}(\omega)$, if $\omega < 1$, or between $\underline{R}$ and $\frac{1}{\beta}$ if $\omega \geq 1$. This occurs when investment opportunities are less scarce; either $\underline{I} \leq I \leq \hat{I}(\omega)$ if $\omega < 1$, or for any $I \geq \underline{I}$ if $\omega \geq 1$. With more investment projects, the total deposit supply is larger and the deposits market clears at a higher interest rate and the liquidity premium is not as large as it is in equilibrium region 0. The bank's choice of k and s follow from their region 1 solution and they will operate all projects. Banks must have some of their own capital invested in the projects ($k > 0$) in order to have enough incentive to supply the effort needed to repay a higher deposit interest rate. But this amount of capital will be less than their total wealth, and they will be able to consume some of their wealth. Buyers will always be liqudity-constrained when banks' wealth is abundant ($\omega < 1$), as discussed earlier. Buyers will also be liquidity-constrained when banks' wealth is scarce because $R < \hat{R}(\omega) < \frac{1}{\beta}$. It is not costless for buyers to hold deposits, so they will not hold enough to afford q^*

- *Equilibrium region 2.* In equilibrium region 2, the interest rate falls between $\hat{R}(\omega)$ and $\min\left\{\bar{R}(\omega), \frac{1}{\beta}\right\}$. This occurs when banks' wealth is relatively scarce ($\omega < 1$) and total investment opportunities are greater than $\hat{I}(\omega)$ (but less that $\bar{I}(\omega)$). Since total investment opportunities are more abundant, the higher deposit supply satisfies buyers' demand for deposits at a higher R. The solution for s, k, and ℓ follow from region 2 of the banks' solution. For I large enough and ω large enough, the upper bound of this region is $R = \frac{1}{\beta}$, and buyers will demand enough deposits to afford q^*. If $(1-\omega)I \geq \beta q^*$, then the amount of deposit funding banks must rely on will be large enough to meet that demand and buyers will not be liquidity-constrained in the DM. If $\omega < \omega_0$, then there can be one more possible equilibrium region.

- *Equilibrium region 3.* In equilibrium region 3, interest rates fall between $\bar{R}(\omega)$ and $\frac{1}{\beta}$. This occurs when banks are the most wealth constrained; wealth is the scarce relative to a large number of investment opportunities, ($\omega < \omega_0$ and $I > \bar{I}(\omega)$). Because total investment is large, the total liquid assets banks can supply will also be relatively large and the liquidity premium on deposits will be smaller. The deposits market clears at an interest rate greater than $\bar{R}(\omega)$, which is the highest interest rate that allows wealth-constrained banks to operate projects with a positive profit. The equilibrium s, k, and ℓ follow from region 3 of the banks' solution. If I is large enough that the total loans banks operate at $R = \frac{1}{\beta}$ can meet buyers' demand for deposits of q^*, even if $\ell < I$, then there will be no liquidity premium.

Note that boundaries do not depend just on banks' wealth W, but on their wealth in proportion to total investment, $\omega = W/I$. This captures how wealth-constrained banks are. When ω is lower and banks are more wealth constrained, the level of investment where they became capital constrained ($\hat{I}(\omega)$) is lower than if they had more wealth for some given I (i.e. a higher ω). If they are less wealth constrained, then it would take a larger level of total investment and a higher resulting equilibrium R before their capital structure is constrained by wealth. Similarly, the cutoff level of total investment ($\bar{I}(\omega)$) and the equilibrium R where they earn zero profits is also lower for lower ω.

5 Equilibrium with CBDC

In this section, I analyze the equilibrium of an economy in which the central bank does issue a CBDC. I will compare equilibrium allocations with CBDC against the benchmark, no-CBDC equilibrium. I will then analze how the central bank's choice of the CBDC interest rate R_d will

affect total welfare. For the following discussion, let the superscript N denote any benchmark, no-CBDC equilibrium values, while the superscript E will denote values from an equilibrium with CBDC.

Because CBDC and deposits are perfect substitutes, buyers will choose to hold the asset that offers the higher rate of return. If the interest rate on CBDC, R_e is set below the no-CBDC equilibrium interest rate on deposits, buyers will have zero demand for CBDC. Therefore if $R_e < R^N(\omega, I)$, there will be no difference in equilibrium allocations in the no-CBDC benchmark. Buyers are indifferent between holding deposits and CBDC if $R_e = R^N(\omega, I)$. But from the first-order condition on deposits (30) and the market-clearing condition (32), buyers' demand for liquid assets is already met by bank deposits, so they do not hold CBDC either. Therefore, buyers only hold CBDC if the interest rate is higher than the interest rate on deposits in the benchmark model, $R_e > R^N(\omega, I)$. This also requires that $R^N < \frac{1}{\beta}$. For a given $\omega > 0$, we can assume that $(1-\omega)(\omega/\omega_0)I < \beta q^*$, so that there will be a liquidity premium in all benchmark equilibrium regions.

Households do not switch entirely to CBDC when $R_e > R^N(\omega, I)$. As some buyers switch, total deposit demand falls, which raises the interest rate in equilibrium. Once $R = R_e$, households are indifferent between the two assets. Since households are holding both assets, (6) and (7) both hold with equality, and in equilibrium, buyers' total liquid asset portfolio satisfies

$$u'(R_e[(1-k^E)\ell^E + e]) = \frac{1}{\beta R_e}. \tag{53}$$

Depending on which region R^N was in and how much of an increase R_e is, the banks' deposit supply $(1-k)\ell$ will either remain the same or lower. If banks are not wealth-constrained in the no-CBDC equilibrium, then a higher R will raise their capital ratio and lower the deposit supply. For example, if R^N is in region 1 and R_e is in region 2, the no-CBDC equilibrium k^N is less than ω and the increase in interest rate lowers deposit supply: $(1-\omega)I < (1-k^N)I$. However the deposit supply may not change if banks are not using any capital investment or if they are wealth-constrained. If R^N was in equilibrium region 2 and R_e still falls in the same region, banks' capital ratio is fixed at ω, so the total deposit supply remains $(1-\omega)I$. If R^N is in equilibrium region 3, an increase in interest rates would also lower total projects operated for a given I.

The remaining equilibrium allocations will be determined by where R_e falls in the banks' solution regions, as described in Lemma 1. The following proposition states these results.

Proposition 4. *There exists a unique equilibrium with digital currency where $R_e > R^N$ and*

the households hold a positive amount of CBDC, $e^E > 0$, total deposits $d^E \leq d^N$, and $q^E > q^N$. Equilibrium s, k, and ℓ satisfy the solution in Lemma 1 for $R = R_e$.

This suggests that introducing a CBDC with a higher interest rate will always improve the efficiency of exchange. When a CBDC is introduced, buyers' liquid asset portfolio does not depend solely on bank deposits. If there are not enough deposits to satisfy the amount of liquid balances they demand at R_e, they can instead hold CBDC. Total liquid asset holdings are larger and buyers can afford more q. However, how bank effort changes relative to the no-CBDC equilibrium will depend on which equilibrium region R^N is in. Effort increases with R in equilibrium regions if banks adjust their capital ratio as R increases, but will decrease if they do not. Thus one potential tradeoff of introducing a CBDC is increasing the riskiness of productive projects.

5.1 Welfare and Optimal Policy

To study whether it will be welfare-improving to introduce a CBDC, I analyze the effect of raising the interest rate on aggregate welfare, depending on the equilibrium region of the benchmark, no-CBDC economy. Since an equilibrium where CBDC is held requires $R_e > R^N$, I will limit attention to the central bank' choice of $R \in [R^N(\omega), \frac{1}{\beta}]$. Aggregate welfare (2) can be written as

$$\mathcal{W}(R) = \{-1 + \beta[s(R)y - c(s(R))]\}\ell(R) + \beta[u(q^E(R)) - q^E(R)] \tag{54}$$

for a given ω and I, eliminating the CM consumption terms that are unaffected by CBDC.

Lemma 2. *In the neighborhood of the no-CBDC equilibrium, introducing a CBDC with $R_e > R^N$ has the following results on q and s:*

1. *DM consumption q always increases.*
2. *Effort s decreases if banks' capital structure is fixed (equilibrium regions 0 and 2), and increases if banks' capital ratio adjusts with R (equilibrium regions 1 and 3).*

Aggregate welfare always increases with an interest-bearing CBDC when banks are not wealth-constrained. For a given $\omega < 1$, there exists a cutoff $\tilde{I}(\omega)$ for which total welfare increases when $I < \tilde{I}(\omega)$.

Raising the interest rate when the benchmark economy is in equilibrium region 0 increases q and lowers s. Banks have no skin-in-the-game in this region and increases in R only reduces

their profits, lowering effort. However, the increased efficiency of exchange by households will have a stronger effect on aggregate welfare. When the economy is in this equilibrium region, deposits are relatively scarce, while banks' project output is relatively low. Therefore, introducing another liquid asset and improving payment eficiency has a greater positive effect on welfare than the negative impact of lower effort on banks' project output.

In equilibrium regions where total lending is not affected by R, the marginal change in aggregate welfare is given by

$$\frac{d\mathcal{W}}{dR} = \beta[y - c'(s)]\frac{ds}{dR} + \beta[u'(q) - 1]\frac{dq}{dR}. \tag{55}$$

If the benchmark economy is in equilibrium region 1, then introducing a CBDC with a higher interest rate increases both q and s. When the CBDC increases the cost of deposits, banks will use more capital financing and thus exert more effort. Since there is a positive effect on both investment payoffs and payment efficiency, total welfare increases.

The effect of a CBDC when the benchmark economy is in equilibrium region 2 or 3 is ambiguous. In equilibrium region 2, banks' capital structure is constrained and like in region 0, increases in R reduces their profits and lowers effort. Since q is increasing with R, there will be an ambiguous effect on total welfare.In equilibrium region 3, banks adjust their capital structure and use more k as R increases, which increases effort. There is a positive effect from both increased s and q on total welfare. However a higher R will also reduce the total number of projects banks can operate. The total quantity of lending will decrease relative to the benchmark economy if the benchmark equilibrium is in region 3. To account for this, the marginal effect of R on aggregate welfare in equilibrium region 3 is given by:

$$\frac{d\mathcal{W}}{dR} = \Big\{ -1 + \beta[sy - c(s)] \Big\}\frac{d\ell}{dR} + \beta(y - c'(s))\ell\frac{ds}{dR} + \beta(u'(q) - 1)\frac{dq}{dR}. \tag{56}$$

If total investment is less than $\tilde{I}(\omega)$, then the total amount of deposits banks can issue is relatively small, which means buyers' will have a greater marginal utility to having access to another unit of payment balances. Introducing CBDC when I is less than this cutoff will mean that the positive effect of introducing CBDC on total welfare will outweigh the negative effects on bank operations. But if the equilibrium interest rate is close to $R = \frac{1}{\beta}$, which implies total investments are abundant, then households already have a relatively large supply of bank deposits and holding a unit of CBDC has a smaller marginal benefit. Then the negative effects on bank outputs will outweigh the positive effect on total welfare.

The central banks' choice of R_e when issuing CBDC will depend on how wealth-constrained

banks are relative to total investment. We will focus our attention on the central banks' choice of $R_e \in [R^N, \frac{1}{\beta}]$. Proposition 5 follows from the Lemma 2.

Proposition 5. *If $\omega \geq 1$, the optimal policy satisfies $R_e = \frac{1}{\beta}$. If $\omega < 1$ and $I < \tilde{I}(\omega)$, then the optimal policy is satisfied for some $R_e \in (R^N, \frac{1}{\beta})$. Otherwise, introducing CBDC is not desirable.*

When $R = \frac{1}{\beta}$, households will hold enough liquid assets for efficient exchange: $u'(q^*) = 1$. When $\omega \geq 1$, banks continue to raise effort and raise their capital ratio as R increases. Then at $R = \frac{1}{\beta}$ is in region 1, $k = 1$ and $s = s^*$ such that $y - c'(s^*) = 0$. As R increases to $R_e = \frac{1}{\beta}$, the marginal increases to investment efficiency and payment efficiency in (55) decrease to zero, and aggregate welfare is maximized when $R_e = \frac{1}{\beta}$. Figure 3 depicts the welfare gains of raising R^N to the Friedman rule compared to the optimal policy, for a given I.[8] As we can see for an abundant wealth case ($\omega = 1.2$), the optimal policy coincides with the Friedman rule.

When $\omega < 1$, the economy will be in equilibrium region 2 or 3 at $R = \frac{1}{\beta}$. In both regions, $d\mathcal{W}/dR$ is negative at $R = \frac{1}{\beta}$ (full details are in the appendix), so the optimal policy will not be the Friedman rule. However, it can still be beneficial to introduce a CBDC that returns some $R_e > R^N$. Let $\tilde{R}(\omega)$ be the no-CBDC equilibrium interest rate that corresponds to $\tilde{I}(\omega)$ defined in Lemma 2. If $R^N < \tilde{R}(\omega)$, then the optimal policy for central banks is to set $R_e = \tilde{R}(\omega)$. If $R^N > \tilde{R}(\omega)$, the central bank should not introducing a CBDC with a higher interest rate, as that will not have a positive effect on aggregate welfare.

Figure 3 illustrates these cases for $\omega = 0.5$ and $\omega = 0.15$. While the equilibrium R^N for $\omega = 0.5$ is initially in region 1, where banks are not wealth constrained, banks will eventually hit the constraint as R increases. The welfare-optimizing $\tilde{R}(0.5)$ is below the Friedman rule. In this case, it will still be beneficial to raise R, but raising the rate beyond $\tilde{R}(0.5)$ causes welfare to decrease as the decrease to effort outweighs the benefit of increased liquidity. When wealth is very scarce, $\omega = 0.15$, Figure 3 shows that raising R to the Friedman rule lowers welfare. In fact, in this example, raising R from the equilibrium rate at all will lower welfare. Without CBDC, the banks' effort and capital are already constrained and raising R will further decrease effort. This suggests that the equilibrium interest rate is greater than the welfare-optimizing R, $R^N(0.15) > \tilde{R}(0.15)$, and since the central bank cannot lower the deposit rate below R^N, the central bank cannot increase welfare by introducing a CBDC.

[8] This version of the graph follows the same functional forms and parameters as previous figures except that $y = 5$ and $\gamma = 5.5$. Total investment I is fixed at 1.

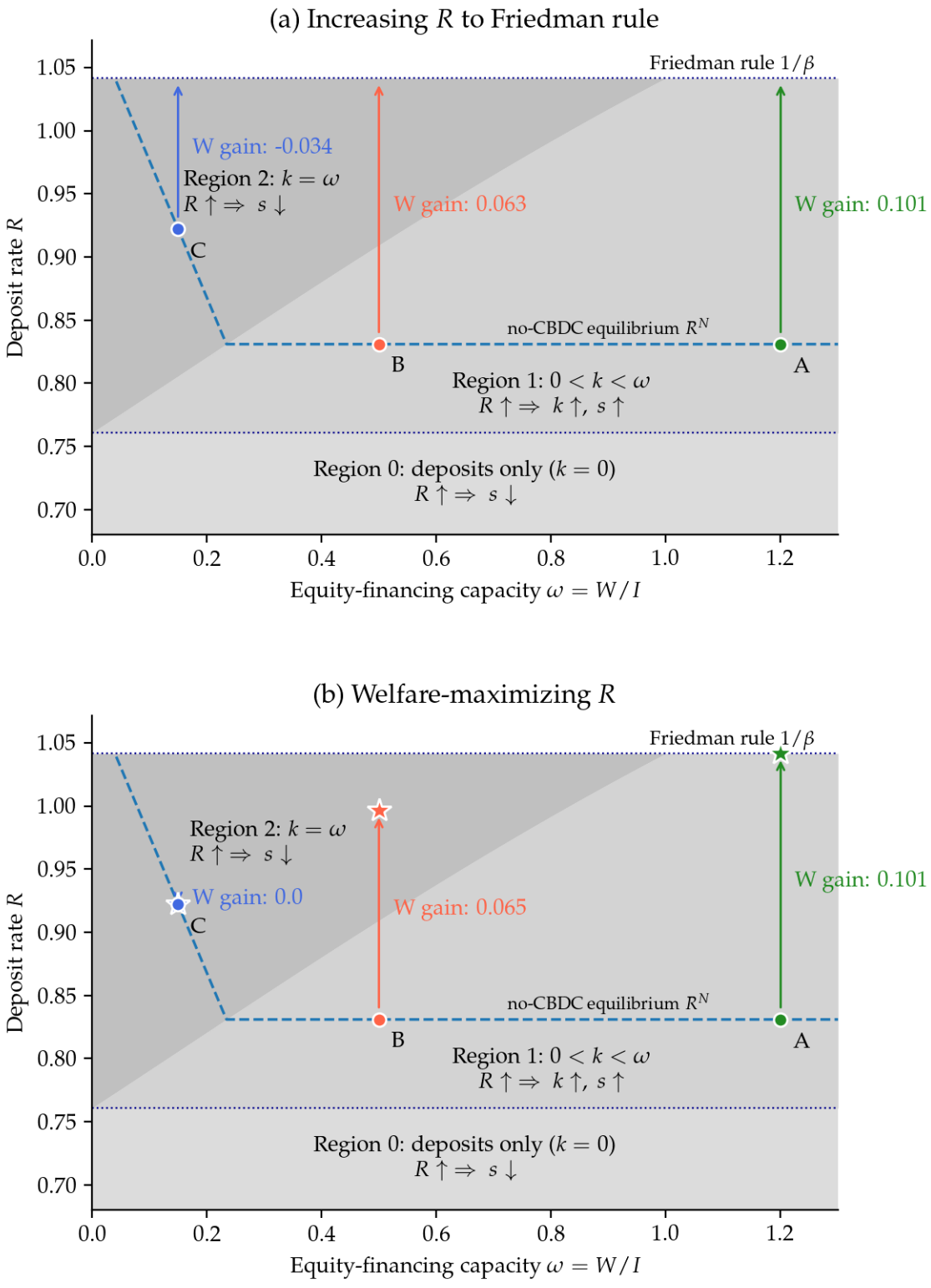


Figure 3: Comparing increases in R_e.

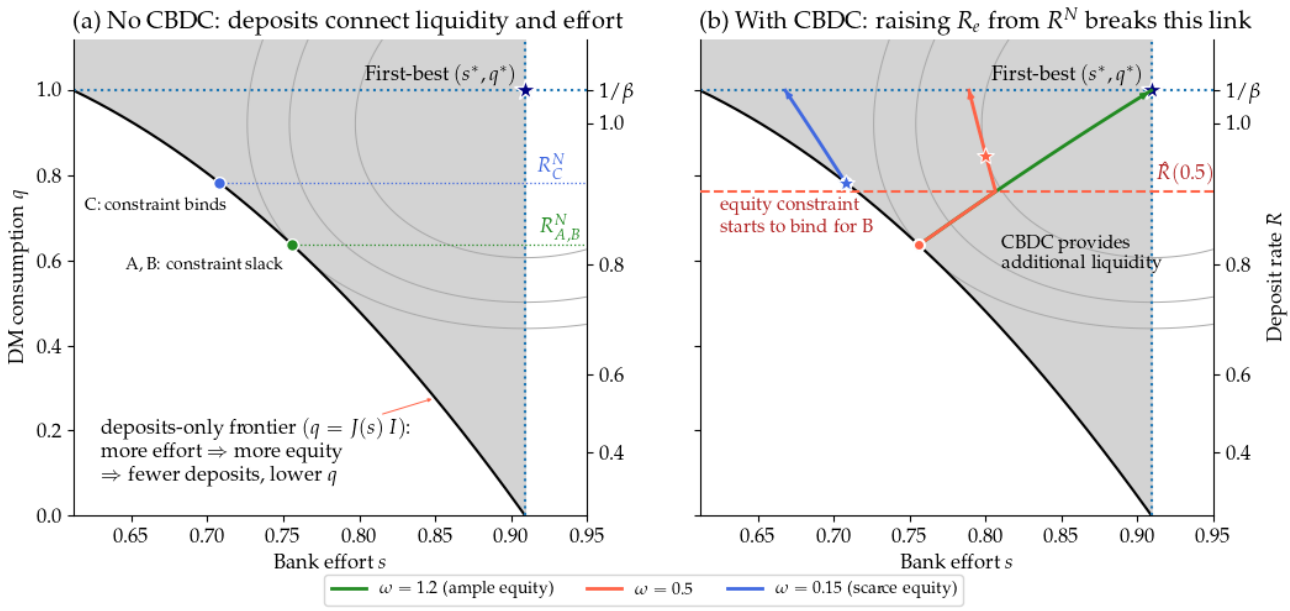


Figure 4: The effects of bank deposit-provision for household liquidity (q) and bank effort (s).

5.2 Discussion

The role of banks. Banks serve two major functions in the economy: (1) they provide deposits, which households can use as payments, and (2) they make loans and fund productive projects. The banks' provision of both roles creates a tension on their risk-taking, as the provision of deposits with a relatively low cost prevents banks from exerting the socially efficient level of effort. This arises when bank deposits are the only medium of exchange. When banks serve both roles, it is not possible to simultaneously achieve q^* or s^*.

Figure 4 depicts this tension. Points A, B, and C correspond to the same points in Figure 3, with the wealth constraint not binding for either A or B in the no-CBDC equilibrium. The amount of deposits the banks supply, which allows buyers to afford q, and bank effort, is connected by $q = R(1-k)I = J(s)I$. Along this deposits-only frontier, the first-best outcome is not possible. At the first-best level of consumption q^*, banks' effort is inefficiently low. But at $s = s^*$, banks would not issue deposits and buyers would not be able to purchase any goods,$q = 0$.

Introducing another form of money (here, CBDC) creates separation between these two roles. With another medium of exchange, it is possible to achieve both optimal payment efficiency and optimal effort. When CBDC is a perfect substitute for deposits in payment and the central bank can choose the interest rate, then implementing the Friedman rule allows households to afford q^*. Since this also raises banks' cost of deposit funding to $R = 1/\beta$, banks become entirely

capital-financed instead, if they have enough wealth to do so. All of the funding is from banks' own wealth, and their skin in the game is at the highest at this point, so banks have an incentive to exert s^*. This is shown by the green arrow in panel (b) of Figure 4. As R increases, both q and s increase towards the first-best levels of DM consumption and effort. But at the first-best, the bank is no longer providing deposits to households. The two bank roles are separated, with the central bank performing the liquidity provision role and banks performing the investment role.

However it is likely that banks face difficulties in raising more capital and must still rely on deposits to fund projects. On one hand, deposit funding allows banks to fund more projects than they could with capital alone. At the same time, because banks' roles cannot completely separate, lending is still exposed to the moral hazard problem created by issuing deposits. When this is the case, the central bank must consider the tradeoffs between payment efficiency and bank investments when setting the optimal policy. As seen by the orange arrow in Figure 4, raising the deposit rate improves both consumption and effort until the banks' wealth constraint starts to bind. Then, while q can increase to q^*, bank effort will begin to decrease after that point and s cannot reach s^*. And when banks' equity is more severly limited (as shown by the blue arrow), raising R only decreases bank efforts.

Disintermediation. In most cases, banks will choose to operate all investment projects available and in those cases, we focus on how CBDC affects the quality of lending via banks' effort. Under certain assumptions and for wealth scarce enough, there may be some cases in which introducing CBDC decreases the quantity of lending. A competing CBDC increases the interest rate on deposits and leads to bank using more capital-financing, but when equity is scarce, the amount of projects the bank can operate at that level of k decreases. Thus a CBDC can disintermediate banks.

This effect on total welfare can be captured by the first term in (56):

$$[-1 + \beta(sy - c(s))]\frac{d\ell}{dR}.$$

The term in the brackets denotes the banks' profits when fully capital financed.There may exist incentive-compatible levels of effort where this profit is negative. This tells us that per-project output y is low enough or the cost of effort $c(s)$ is high enough that it would not be profitable to run the projects with only capital financing. Banks can operate these projects with at least zero profits if they can use relatively inexpensive deposit funding. In this case, disintermediating banks out of operating these projects can improve total welfare. Otherwise, if projects are always profitable to run, even with full capital financing, then this disintermediation has a

negative effect on total welfare.

6 Conclusion

In this study, I present a model that examines how the usage of bank deposits as a means of payment interacts with bank capital structure and risk-taking decisions. I then analyze how the introduction of another means of payment, like a digital currency, impacts those bank decisions.

By combining a model of money following the New Monetarist framework with a model of banks that includes a risk-taking margin, I can show how the provision of deposits prevents banks from exerting the socially efficient level of effort. Because banks have limited liability and do not have to repay depositors when projects fail, the riskiness of bank lending is highest when banks are the most deposit-financed. They will only exert the efficient level of effort when they are completely capital financed. Because households need deposits to pay for consumption, banks will always provide deposits in equilibrium, and thus never exert this efficient level of effort.

The introduction of a CBDC, bearing an interest rate that the central bank can set, can allow for more efficient payments between buyers and sellers. The effect of a CBDC on banks' effort choices will depend on how equity-constrained the banks may be. If banks do not face any constraints raising more capital, running the Friedman rule through the CBDC interest rate achieves the first-best allocation for both payments and bank effort. However, if banks do face equity constraints, then the central bank must consider tradeoffs between improving the efficiency of payments and reducing either the quality or quantity of loans. It can still be optimal to raise the interest rate through CBDC, but running the Friedman rule is not optimal in this case.

The features of this model are highly stylized, and there are further aspects of the banking sector that are not included in this model. Future research could include more detailed analysis on bank market power or on the effects of aggregate risk. CBDC is modeled relatively broadly here, which allows it to be applied to other new forms of digital payments, but there may be features of CBDC design (i.e. holding limits, acceptability by sellers) that could impact the results.

These findings contribute to our understanding of how innovations in digital payments can affect bank lending and how that affects policy decisions. A CBDC could be a useful tool for disciplining bank risk, but policymakers must take into account banks' capital when deciding on the optimal policy.

7 Appendix

A Proofs

A.1 Proof of Lemma 1

Proof. The bank's problem is:

$$\max_{s,k,\ell} -k\ell + \beta[sy - R(1-k) - c(s)]\ell \tag{A.1}$$

subject to the bank's incentive constraint

$$R(1-k)\ell \leq s(y - c'(s))\ell, \tag{A.2}$$

their wealth constraint

$$k\ell \leq W, \tag{A.3}$$

the upper limit on total loans

$$\ell \leq I, \tag{A.4}$$

and other feasibility constraints on k and ℓ

$$k \geq 0 \tag{A.5}$$

$$k \leq 1 \tag{A.6}$$

$$\ell \geq 0 \tag{A.7}$$

Let λ_1, λ_2, and λ_3 be the Lagrange multipliers on the first three constraints, respectively, and let μ_1, μ_2, and μ_3 be the Lagrange multipliers on the last three feasibility constraints, respectively. Define $\omega = W/I$ as the wealth to investment ratio.

We first simplify the first-order conditions to the bank's problem. Then we construct the regions in the lemma and derive the properties of the solution in each region.

Preliminaries:

Define s^* as the solution to $y - c'(s) = 0$ and $s^\dagger$ as the solution to $sc''(s) + c'(s) - y = 0$. By the assumptions on $c(s)$ (see Assumption 1), if $s = 0$, $sc''(s) + c'(s) = 0$. If $s = s^*$, $s^*c''(s^*) + c'(s^*) > y$. Then by the intermediate value theorem, there exists an $s^\dagger \in (0, s^*)$ such that $sc''(s) + c'(s) = y$. The derivative of $sc''(s) + c'(s)$ with respect to s is $sc'''(s) + 2c''(s) > 0$,

so since this function is monotonically increasing, $s^\dagger$ is also unique. Next we can prove s^* exists and is unique. The function $c'(s)$ is monotonically increasing in s, which implies for all $s > s^*$, $c'(s) > c'(s^*) = y$. At $s^\dagger$, $c'(s^\dagger) = y - s^\dagger c''(s^\dagger) < y$. By the intermediate value theorem, s^* also exists and is unique. This also implies $s^\dagger < s^*$. By assumption on $c(s)$, both $s^\dagger$ and s^* are unique and $0 < s^\dagger < s^*$.

Next, we define the functions

$$J(s) = s(y - c'(s)), H(s) = \frac{y - c'(s)}{sc''(s) + c'(s) - y}$$

By the assumptions on $c(s)$, the function $J(s)$ satisfies $J'(s) > 0$ for $0 < s < s^\dagger$, $J'(s) < 0$ for $s^\dagger < s < s^*$, and $J''(s) < 0$. $J(s) = 0$ at $s = 0$ and $s = s^*$, and is maximized at $s = s^\dagger$.

The function $H(s)$ is downward sloping

$$H'(s) = \frac{-c''(s)[sc''(s) + c'(s) - y] - (y - c'(s))[sc'''(s) + 2c''(s)]}{(sc''(s) + c'(s) - y)^2} < 0,$$

asymptotes to infinity at $s \to s^\dagger$, and equals 0 at $s = s^*$. It will be clear below that the solution to the bank's problem must have $s \in (s^\dagger, s^*]$, so the properties of H and J in this interval are the relevant ones.

First-order conditions:

$$0 = \beta[y - c'(s)]\ell + \lambda_1[y - c'(s) - sc''(s)]\ell \tag{A.8}$$

$$0 = (\beta R - 1)\ell + \lambda_1 R\ell - \lambda_2\ell + \mu_1 - \mu_2 \tag{A.9}$$

$$0 = \beta[sy - c(s) - R(1-k)] - k + \lambda_1[s(y - c'(s)) - R(1-k)] - \lambda_2 k - \lambda_3 + \mu_3 \tag{A.10}$$

Simplifying the problem: First, note that (A.2) always holds with equality. Suppose not, and that $R(1-k)\ell < s(y - c'(s))\ell$. By complementary slackness, $\lambda_1 = 0$ and (A.8) becomes $0 = \beta[y - c'(s)]\ell$. But this would mean $R(1-k)\ell < 0$, which is a contradiction for any $R, \ell \geq 0$ and $k \in [0, 1]$. We can rearrange (A.8) as

$$\lambda_1 = \frac{\beta[y - c'(s)]\ell}{[sc''(s) + c'(s) - y]\ell} = \beta H(s).$$

We make two simplifying observations. Firs, under the assumptions that we make, $k < 1$ unless $R \geq \frac{1}{\beta}$ and $\ell > 0$.

- If $k = 1$, the incentive constraint is $0 = s(y - c'(s))\ell$ and either $s = 0$ or $s = s^*$. If $s = 0$, then $\lambda_1 = -\beta$, which is not possible by nonnegativity constraints on Lagrange multipliers. Then suppose $s = s^*$ and $\lambda_1 = 0$. Since $k = 1$, $k > 0$ and $\mu_1 = 0$. The FOC (A.9) can be simplified as $\mu_2 + \lambda_2\ell = (\beta R - 1)\ell$. Due to the nonnegativity constraints on the multipliers, this implies that $(\beta R - 1)\ell \geq 0$, which is only possible for $R \geq \frac{1}{\beta}$ when $\ell > 0$.

- Suppose $\ell = 0$. Then the bank is maximizing a value equal to zero, according to (A.1). However, by Assumption 2, there exists a combination of s, k, and ℓ that achieves a strictly positive objective value (namely, $k = 1$, $\ell = \min\{W, I\}$, and $s = s^*$). But in that case, $\ell = 0$ cannot be maximizing profits and would not be the solution.

Therefore, we will assume that $k < 1$ and $\ell > 0$ for the following analysis and that $\mu_2 = \mu_3 = 0$.

With these simplifications, we can rewrite the first-order conditions as

$$\lambda_1 = \beta H(s) \tag{A.11}$$

$$\frac{1}{\beta R} - 1 = H(s) - \frac{\lambda_2}{\beta R} + \frac{\mu_1}{\beta R \ell} \tag{A.12}$$

$$\lambda_3 = \beta[sy - c(s) - R(1 - k)] - k - \lambda_2 k \tag{A.13}$$

Second, either (A.3) or (A.4) must hold with equality. Suppose that neither (A.3) or (A.4) hold with equality, so $\ell < I$ and $W > k\ell$, and subsequently $\lambda_2 = \lambda_3 = 0$ by complementary slackness. By (A.13), $0 = \beta[sy - c(s) - R(1-k)] - k$ and according to (A.1), the bank's objective value equals zero. But Assumption 2, the bank can choose $k = 1$, $\ell = \min\{W, I\}$, and $s = s^*$ and obtain a strictly positive objective value. If $W < I$, tbe bank could always increase k and ℓ until $k = 1$ and $k\ell = W$ and obtain positive profits. Similarly, if $W \geq 1$, the bank could choose to increase ℓ to I and make positive profits. Therefore $\ell < I$ and $W > k\ell$ together cannot be a solution. This tells us the solution must fall into one of 3 cases: either (A.3), (A.4), or both hold with equality.

The case with $\omega \geq 1$:

If $\omega > 1$, then (A.3) does not bind with equality ($W > I \geq k\ell$) and $\lambda_2 = 0$. Then by (A.13), $\lambda_3 > 0$ and $\ell = I$. The solution will fall into the case where only (A.4) binds with equality. We only need to distinguish 2 further cases depending on whether (A.5) binds.

Let $\underline{R}$ and $\underline{s}$ be the solution to the system

$$\underline{R} = J(\underline{s}) \tag{A.14}$$

$$\frac{1}{\beta \underline{R}} - 1 = H(\underline{s}) \tag{A.15}$$

By substituting $\underline{R}$ into the (A.15), we see the required $\underline{s}$ solves

$$\frac{1}{\beta J(\underline{s})} - H(\underline{s}) = 1.$$

Let $f(s) \equiv \frac{1}{\beta J(s)} - H(s)$. The function asymptotes to $-\infty$ as $s \to s^\dagger$ and $f(s^*)$ asymptotes to $+\infty$. The function is also monotonically increasing:

$$f'(s) = -\frac{1}{\beta J(s)^2} J'(s) - H'(s) > 0.$$

Since $f(s^\dagger) < 1 < f(s^*)$, then by the intermediate value theorem, a solution $\underline{s} \in (s^\dagger, s^*)$ exists and is unique by the properties of J and H. As discussed earlier, $J(s)$ is monotonically decreasing between $s^\dagger$ and s^*, so $\underline{R} = J(\underline{s})$ is also uniquely defined. Additionally, H is decreasing in s, which means $H(\underline{s}) > H(s^*) = 0$ and by (A.15), $\underline{R} < \frac{1}{\beta}$.

We prove that (A.5) binds if and only if $R < \underline{R}$.

- First suppose not and that $R > \underline{R}$ when $k = 0$. When $k = 0$, $R = J(s)$ by (A.2). If $R > \underline{R}$, then $J(s) > J(\underline{s})$ and this implies $s < \underline{s}$. We also have

$$\frac{1}{\beta R} - 1 = H(s) + \frac{\mu_1}{\beta R \ell}$$
$$\frac{1}{\beta \underline{R}} - 1 = H(\underline{s})$$

  If $R > \underline{R}$, then $\frac{1}{\beta R} - 1 < \frac{1}{\beta \underline{R}} - 1$ and therefore $H(s) + \frac{\mu_1}{\beta R \ell} < H(\underline{s})$. Because H is a decreasing function and $\mu_1 \geq 0$, this implies that s is strictly larger than $\underline{s}$. But this is a contradiction.

- To prove the other direction, suppose that $R < \underline{R}$ and k is instead greater than 0. Then $R(1-k) < \underline{R}$ and $J(s) < J(\underline{s})$, which means that $s > \underline{s}$. When $R < \underline{R}$,

$$\frac{1}{\beta R} - 1 > \frac{1}{\beta \underline{R}} - 1$$

and $H(s) > H(\underline{s})$ ($\mu_1 = 0$ since we assumed $k > 0$). This implies that $s < \underline{s}$, which is a contradiction.

Therefore, if $R < \underline{R}$, then $k = 0$.

If $\omega = 1$, λ_2 also equals 0. (When $R < \frac{1}{\beta}$, $k < 1$ and $W = I > k\ell$. If $R = \frac{1}{\beta}$ and $k = \omega = 1$, then by (A.12), $\lambda_2 = \mu_1/\ell$. Since $k > 0$, $\mu_1 = 0$ and $\lambda_2 = 0$.) Since $\lambda_2 = 0$, then λ_3 is also positive. Under the simplifying assumption that $k < 1$, (A.3) will not hold with equality, and we can apply the same analysis given above. The results would still apply if $k = 1$; the only difference would be that both (A.3) and (A.4) hold with equality.

In this region, the solution falls into 2 cases.

1. If $R \leq \underline{R}$, $k = 0$, $\ell = I$, and s satisfies $R = J(s)$.

2. If $R > \underline{R}$, the bank uses some capital ($k > 0$). In that case, s satisfies

$$\frac{1}{\beta R} - 1 = H(s),$$

   k satisfies $R(1-k) = J(s)$ and $\ell = I$.

The case with $0 < \omega < 1$:

Defining the bounds: Let $\underline{R}$ and $\underline{s}$ be, as above, the solution to

$$\underline{R} = J(\underline{s}) \tag{A.16}$$

$$\frac{1}{\beta \underline{R}} - 1 = H(\underline{s}) \tag{A.17}$$

Let $\hat{R}(\omega)$ and $\hat{s}(\omega)$ be the solution to

$$\hat{R}(1-\omega) = J(\hat{s}) \tag{A.18}$$

$$\frac{1}{\beta \hat{R}} - 1 = H(\hat{s}) \tag{A.19}$$

and let $\bar{R}(\omega)$ and $\bar{s}(\omega)$ be the solution to

$$\bar{R}(1-\omega) = J(\bar{s}) \tag{A.20}$$

$$\bar{s}y - c(\bar{s}) - \bar{R} + H(\bar{s})[J(\bar{s} - \bar{R})] = 0 \tag{A.21}$$

The last two sets of bounds depend on the bank's given wealth to investment ratio, but for ease of notation, I drop the ω in the following analysis.

We first prove that the bounds are well-defined and $\underline{R} \leq \hat{R} < \bar{R}$, with the first inequality strict if $\omega > 1$. We proved earlier that $\underline{s}$ exists and is unique, as is $\underline{R}$. Now let $f(s) \equiv \frac{1-\omega}{\beta J(s)} - H(s)$. This function asymptotes to $-\infty$ as $s \to s^\dagger$ and to $+\infty$ as $s \to s^*$ and is monotonically increasing:

$$f'(s) = -\frac{1-\omega}{\beta J(s)^2} J'(s) - H'(s) > 0.$$

Then since $f(s^\dagger) < 1 < f(s^*)$, there exists a unique $\hat{s} \in (s^\dagger, s^*)$ such that $f(s) = 1$ by the intermediate value theorem. Since we have

$$\frac{1}{\beta J(\underline{s})} - H(\underline{s}) = 1$$
$$\frac{1}{\beta J(\hat{s})} - H(\hat{s}) = 1$$

by definition, then for $\omega > 0$, either $J(\hat{s}) < J(\underline{s})$ or $H(\hat{s}) < H(\underline{s})$ needs to hold, which both require that $\hat{s} > \underline{s}$.

Next, we can also use the intermediate value theorem to prove a unique solution $\bar{s} \in (s^\dagger, s^*)$ exists and that $\bar{s} < \hat{s}$. Using $\bar{R} = \frac{J(\bar{s})}{1-\omega}$, we can rewrite (A.21) as

$$\bar{s}y - c(\bar{s}) - \frac{J(\bar{s})}{1-\omega}[1 + H(\bar{s})\omega] = 0$$

Let $g(s) \equiv sy - c(s) - \frac{J(s)}{(1-\omega)}[1 + H(s)\omega]$. At $s^\dagger$, $g(s)$ asymptotes to $-\infty$ since $H(s^\dagger) \to \infty$. At s^*, $J(s^*) = 0$ and $s^*y - c(s^*) > 0$ by Assumption 2, so $g(s^*) > 0$. Therefore, there exists some $\bar{s} \in (s^\dagger, s^*)$ such that $g(\bar{s}) = 0$. Since $g(s)$ is increasing in s, $\bar{s}$ is also unique.

$$g'(s) = y - c'(s) - \frac{J'(s)}{1-\omega}[1 + H(s)\omega] - \frac{J(s)}{1-\omega}[H'(s)\omega] > 0$$

To prove that $\bar{s} < \hat{s}$, suppose the contrary. Since $g'(s) > 0$, this implies that $g(\hat{s}) < 0$. Note that from substituting λ_2 from (A.12) into (A.13) when $k = \omega$:

$$\lambda_3 = \beta\{sy - c(s) - R[1 + H(s)\omega]\}$$

which is the bank's profits when wealth-constrained. If $g(\hat{s}) < 0$, this means that $\lambda_3 < 0$ or the bank profits are negative, which cannot be a solution. Therefore, $\bar{s} < \hat{s}$.

Having established $\underline{s} < \hat{s}$ and $\bar{s} < \hat{s}$, we can also prove that $\underline{R} < \hat{R} < \bar{R}$. Since H is

decreasing in s, $H(\underline{s}) > H(\hat{s})$ and (A.17) and (A.19) imply that $\underline{R} < \hat{R}$. Next, because J is decreasing, $J(\hat{s}) < J(\bar{s})$ and by the definitions in (A.18) and (A.20), $\hat{R} < \bar{R}$.

We also prove the following:

1. If $J(s^\dagger) \geq \frac{1}{\beta}$, then $\bar{R} > \frac{1}{\beta}$.
2. If $J(s^\dagger) < \frac{1}{\beta}$, then there exists $\omega_0 \in (0,1)$ such that $\bar{R} < \frac{1}{\beta}$ if and only if $\omega < \omega_0$.

First suppose $J(s^\dagger) \geq \frac{1}{\beta}$. Then there exists some $s^{\dagger\dagger} \in (s^\dagger, s^*)$ such that $J(s) = \frac{1}{\beta}$. Let

$$\Pi(s) \equiv sy - c(s) - \frac{1}{\beta} + H(s)\left[J(s) - \frac{1}{\beta}\right].$$

By assumption, $J(s^\dagger) > \frac{1}{\beta}$ and we know that $J(s^*) = 0$ and that J is decreasing between $s^\dagger$ and s^*, so there exists a unique solution $s^{\dagger\dagger}$ such that $J(s^{\dagger\dagger}) = \frac{1}{\beta}$.

When $J(s^\dagger) > \frac{1}{\beta}$, $\Pi(s)$ i always greater than 0.

$$\Pi'(s) = y - c'(s) + H(s)\left[J(s) - \frac{1}{\beta}\right] + H(s)J'(s)$$

Note that $H(s)J'(s) = -(y - c'(s))$ so this expression simplifies to

$$\Pi'(s) = H'(s)\left[J(s) - \frac{1}{\beta}\right].$$

Profits decrease between $(s^\dagger, s^{\dagger\dagger})$, when $J(s) - \frac{1}{\beta} > 0$. At $s^{\dagger\dagger}$, $\Pi'(s) = 0$ and $\Pi(s^{\dagger\dagger})$ by the convexity of $c(s)$:

$$\begin{aligned}
\Pi(s^{\dagger\dagger}) &= s^{\dagger\dagger}y - c(s^{\dagger\dagger}) - \frac{1}{\beta} \\
&= s^{\dagger\dagger}y - c(s^{\dagger\dagger}) - (s^{\dagger\dagger}y - s^{\dagger\dagger}c'(s^{\dagger\dagger})) \\
&= s^{\dagger\dagger}c'(s^{\dagger\dagger}) - c(s^{\dagger\dagger}) \\
&> 0
\end{aligned}$$

Profits increase between $(s^{\dagger\dagger}, s^*)$, where $J(s) - \frac{1}{\beta} < 0$, to $\Pi(s^*)$, which is positive by assumption:

$$\Pi(s^*) = s^*y - c(s^*) - \frac{1}{\beta} > 0.$$

Therefore, for all $s \in (s^\dagger, s^*)$, $\Pi(s) > 0$.

We can then show that $\bar{R} > \frac{1}{\beta}$. Suppose not, and that $\bar{R} \leq \frac{1}{\beta}$. It follows that

$$\begin{aligned} 0 &= \bar{s}y - c(\bar{s}) - \bar{R} + H(\bar{s})[J(\bar{s}) - \bar{R}] \\ &\geq \bar{s}y - c(\bar{s}) - \frac{1}{\beta} + H(\bar{s})\left[J(\bar{s}) - \frac{1}{\beta}\right] \\ &> 0 \end{aligned}$$

which is a contradiction.

Next, if $J(s^\dagger) < \frac{1}{\beta}$, then there exists an s_0 such that $\Pi(s) = 0$:

$$s_0 y - c(s_0) - \frac{1}{\beta} + H(s_0)\left[J(s_0) - \frac{1}{\beta}\right] = 0.$$

From properties of H, $\Pi(s^\dagger) \to -\infty$ and $\Pi(s^*) > 0$. Therefore, by the intermediate value theorem, there exists an s_0 such that $\Pi(s_0) = 0$. Since J is decreasing and $J(s^\dagger) < \frac{1}{\beta}$ here by assumption,

$$\Pi'(s) = H'(s)\left[J(s) - \frac{1}{\beta}\right] > 0$$

profits are always decreasing between $s^\dagger$ and s^*. This also proves s_0 is unique.

Let ω_0 be such that $\frac{1}{\beta}(1 - \omega_0) = J(s_0)$. We can show that $\bar{R} < \frac{1}{\beta}$ if and only if $\omega < \omega_0$.

- Suppose that $\omega < \omega_0$ but $\bar{R} \geq \frac{1}{\beta}$. Then

$$\begin{aligned} 0 &= \bar{s}y - c(\bar{s}) - \bar{R} + H(\bar{s})[J(\bar{s}) - \bar{R}] \\ &< \bar{s}y - c(\bar{s}) - \frac{1}{\beta} + H(\bar{s})\left[J(\bar{s}) - \frac{1}{\beta}\right] \equiv \Pi(\bar{s}) \end{aligned}$$

  As defined earlier, $\Pi(s_0) = 0$. If $\Pi(\bar{s}) > 0$ and $\Pi'(s) > 0$, this implies that $s_0 < \bar{s}$. At the same time,

$$J(s_0) = \frac{1}{\beta}(1 - \omega_0) < \bar{R}(1 - \omega) = J(\bar{s})$$

  which implies that $\bar{s} < s_0$. However, this is a contradiction with the previous result.

- Suppose that $\bar{R} < \frac{1}{\beta}$ but $\omega > \omega_0$. In this case

$$\begin{aligned} 0 &= \bar{s}y - c(\bar{s}) - \bar{R} + H(\bar{s})[J(\bar{s}) - \bar{R}] \\ &> \bar{s}y - c(\bar{s}) - \frac{1}{\beta} + H(\bar{s})\left[J(\bar{s}) - \frac{1}{\beta}\right] \equiv \Pi(\bar{s}). \end{aligned}$$

Then if $\Pi(s_0) = 0$ and $\Pi(\bar{s}) < 0$, this implies $s_0 > \bar{s}$. But from $\frac{1}{\beta}(1-\omega_0) > \bar{R}(1-\omega)$, this means $J(s_0) > J(\bar{s})$ and $s_0 < \bar{s}$, which is also a contradiction.

Therefore $\bar{R} < \frac{1}{\beta}$ if and only if $\omega < \omega_0$.

Characterizing the solution: For a given $\omega > 0$, the bank's solution falls into four possible cases depending on R, which it takes as given.

1. *If $R < \underline{R}$, then* (A.5) *and* (A.4) *hold with equality and* (A.3) *does not bind. The solution is given by $k = 0$, $\ell = I$, and s that satisfies $R = J(s)$.*

   As proved earlier, $k = 0$ if and only if $R < \underline{R}$. Then since the bank's IC (A.2) always holds with equality, s must satisfy $R = J(s)$. To verify that $\ell = I$,

   $$\begin{aligned}\lambda_3 &= \beta[sy - c(s) - R] \\ &= \beta[sy - c(s) - s(y - c'(s))] \\ &= \beta[sc'(s) - c(s)] \\ &> 0\end{aligned}$$

   and by complementary slackness, $\ell = I$ must hold with equality. Alternatively, because we know either (A.3) or (A.4) must hold with equality and (A.3) does not in this case ($W > k\ell = 0$), then it must be (A.4) holding with equality and $\ell = I$.

   Also, given $R < \underline{R}$, $J(s) < J(\underline{s})$, $s > \underline{s}$, and $H(s) < H(\underline{s})$. We know that $\frac{1}{\beta J(\underline{s})} - H(\underline{s}) = 1$, so $\frac{1}{\beta J(s)} - H(s) > 1$ or

   $$\frac{1}{\beta R} - 1 > H(s).$$

2. *If $\underline{R} < R < \hat{R}$, then* (A.4) *holds with equality and* (A.3) *does not bind. Effort s solves*

   $$\frac{1}{\beta R} - 1 = H(s),$$

   optimal k is interior and solves $R(1-k) = J(s)$, and $\ell = I$.

   Because $R > \underline{R}$, k must be larger than 0. To verify that $W < k\ell$, suppose not and that $W = k\ell$, so that $k = W/\ell \geq \omega$. Then $R(1-k) < \hat{R}(1-\omega)$ and $J(s) < J(\hat{s})$. But this implies $s > \hat{s}$ when it should be that $\underline{s} < s < \hat{s}$ when $\underline{R} < R < \hat{R}$. Therefore, (A.3) does not hold with equality and $\ell = I$.

Since $W > k\ell$ and $k > 0$, $\lambda_2 = \mu_1 = 0$ and by (A.12)

$$\frac{1}{\beta R} - 1 = H(s)$$

in this region, which defines s for a given $R \in (\underline{R}, \hat{R})$. From that s, we then can define k from the IC, $R(1-k) = J(s)$. Since $s < s$, $R < \frac{1}{\beta}$ and $k < 1$, so k is interior.

3. *If* $\hat{R} < R < \min\{\bar{R}, 1/\beta\}$*, then* (A.3) *and* (A.4) *both hold with equality. The solution for* k *is given by* $k = \omega$*, and all loans operate* $\ell = I$*. Effort* s *satisfies* $R(1-\omega) = J(s)$*.*

   To verify that $W = k\ell$, suppose not and $W > k\ell$. Then $\lambda_2 = 0$ and $\frac{1}{\beta R} - 1 = H(s)$. However, if $R > \hat{R}$, then $\frac{1}{\beta R} - 1 < \frac{1}{\beta \hat{R}} - 1$ or $H(s) < H(\hat{s})$ and therefore $s > \hat{s}$. But this means that $J(s) < J(\hat{s})$ and by (A.18) and (A.20), $R < \hat{R}$, which is a contradiction.

   To verify that $\ell = I$, first note that s is decreasing in R when $R(1-\omega) = J(s)$:

$$\frac{ds}{dR} = \frac{1-\omega}{J'(s)} < 0.$$

   Then note that $\lambda_2 > 0$ in this case. When $R > \hat{R}$, $J(s) > J(\hat{s})$ and $s < \hat{s}$. It follows that

$$H(s) > H(\hat{s}) = \frac{1}{\beta R} - 1$$

   which for interior k implies that $\lambda_2 > 0$, according to (A.12). Therefore the first-order condition for ℓ can be written as $\lambda_3 = \beta[sy - c(s) - R + H(s)(J(s) - R)]$. The right-hand side is decreasing in R:

$$\{H'(s)[J(s) - R]\}\frac{ds}{dR} - (1 + H(s)) < 0.$$

   Therefore, for $R < \bar{R}$, $\lambda_3 > 0$ and $\ell = I$.

   If $\bar{R} > \frac{1}{\beta}$ then the upper bound of this region is $R = \frac{1}{\beta}$ and $\lambda_3 = s^* y - c(s^*) - \frac{1}{\beta}$, which is positive by assumption, so $\lambda_3 = 0$ and $\ell = I$ as well.

4. *If* $J(s^\dagger) < 1/\beta$ *and* $\bar{R} < R < 1/\beta$*, then* (A.3) *holds with equality and* (A.4) *does not bind. The solution is given by a level of effort* s *that satisfies*

$$sy - c(s) - R + H(s)[J(s) - R] = 0, \tag{A.22}$$

   a capital ratio k *such that* $R(1-k) = J(s)$*, and loans* $\ell = (\omega I)/k < I$*.*

To verify that $\ell < I$, suppose instead that $\ell = I$. If $W = k\ell$, then $k = \omega$ and $R(1-\omega) = J(s)$. For $R > \bar{R}$, $R(1-\omega) > \bar{R}(1-\omega)$ tells us $J(s) > J(\bar{s})$ and $s < \bar{s}$. We previously showed that $g'(s) > 0$, so if $g(\bar{s}) = 0$, this implies that $g(s) < 0$. However, this means that $\lambda_3 < 0$, which is not possible. Therefore, $\ell < I$ and since at least one of (A.3) or (A.4) must hold with equality, this verifies that $W = k\ell$.

We can also verify $W = k\ell$ through the following: Suppose that $W > k\ell$ and $\lambda_2 = 0$, so that $\frac{1}{\beta R} - 1 = H(s)$. Again since at least one of (A.3) or (A.4) must hold with equality, $\ell = I$ and $W > kI$ or $\omega > k$. Then $J(s) = R(1-k) > \bar{R}(1-\omega) = J(\bar{s})$ and $s < \bar{s}$. But for $R > \bar{R} > \hat{R}$,

$$\frac{1}{\beta R - 1} < \frac{1}{\beta \hat{R}} - 1$$

and $H(s) < H(\hat{s})$, so $s > \hat{s}$, which is a contradiction since s cannot be both less than $\bar{s}$ and greater than $\hat{s}$ when $\bar{s} < \hat{s}$. Therefore $W > k\ell$ is not possible when $R > \bar{R}$.

Then for a given R in this region, s satisfies $sy - c(s) - R + H(s)[J(s) - R] = 0$. As R increases in this region, effort increases so that the equation still equals 0:

$$\frac{ds}{dR} = \frac{1 + H(s)}{H'(s)[J(s) - R]} > 0$$

(this equation cannot fall below zero, otherwise $\lambda_3 < 0$ and the bank's profits are negative.) Then from this s, the IC determines k, $R(1-k) = J(s)$. Lastly, ℓ is determined by k and the bank's given ω and I: $W = k\ell$ or $\ell = W/k = (\omega I)/k$.

I will refer to these regions as equilibrium regions 0, 1, 2, and 3 respectively.

The case with $\omega = 0$:

When $\omega = 0$, then $W = k\ell = 0$ and since $\ell = 0$ cannot be a solution, k is always equal to zero in this case. We still have the bounds $\underline{R}$ and $\underline{s}$ as defined earlier. Since $\omega = 0$, $\underline{R} = \hat{R}$ and $\underline{s} = \hat{s}$ and Region 1, the second case above where $\underline{R} < R < \hat{R}$, does not exist at $\omega = 0$.

If $J(s^\dagger) < \frac{1}{\beta}$, then let $\bar{R}_0 = J(s^\dagger)$. When $k = 0$, the highest level of effort that the bank will provide for $R > 0$ is $s^\dagger$. This is lower bound of the Region 3 listed in the $0 < \omega < 1$ case. For $R > \bar{R}_0$, the bank does not operate.

Boundary given ω: The boundary between region 0 and 1 is independent of wealth, since the bank does not use any capital.

The boundary of region 1 and 2 is given by (A.18) and (A.19).

$$(1-\omega)\frac{d\hat{R}}{d\omega} - \hat{R} = J'(\hat{s})\frac{d\hat{s}}{d\omega}$$
$$-\frac{1}{\beta\hat{R}^2}\frac{d\hat{R}}{d\omega} = H'(\hat{s})\frac{d\hat{s}}{d\omega}$$

In matrix notation, the system of equations can be written as

$$\begin{bmatrix} (1-\omega) & -J'(\hat{s}) \\ -\frac{1}{\beta\hat{R}^2} & H'(\hat{s}) \end{bmatrix} \begin{bmatrix} \frac{d\hat{R}}{d\omega} \\ \frac{d\hat{s}}{d\omega} \end{bmatrix} = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$$

The determinant is

$$\Delta = (1-\omega)H'(\hat{s}) + J'(\hat{s})\frac{1}{\beta\hat{R}} < 0$$

so we know there is a unique solution. By Cramer's rule, we can prove that both $\hat{s}$ and $\hat{R}$ are increasing with wealth.

$$\Delta_R = \begin{vmatrix} \hat{R} & -J'(\hat{s}) \\ 0 & H'(\hat{s}) \end{vmatrix} = \hat{R}H'(\hat{s}) > 0$$

and

$$\Delta_s = \begin{vmatrix} (1-\omega) & \hat{R} \\ \frac{1}{\beta\hat{R}^2} & 0 \end{vmatrix} = -\frac{1}{\beta\hat{R}} < 0$$

$$\frac{d\hat{R}}{d\omega} = \frac{\Delta_R}{\Delta} > 0$$
$$\frac{d\hat{s}}{d\omega} = \frac{\Delta_s}{\Delta} > 0$$

Similarly, the boundary of Region 2 and 3 is given by (A.20) and (A.21). Taking the derivative with respect to ω yields

$$\begin{bmatrix} (1-\omega) & -J'(\bar{s}) \\ -(1+H(\bar{s})) & H'(\bar{s}[J(\bar{s})-\bar{R}]) \end{bmatrix} \begin{bmatrix} \frac{d\bar{R}}{d\omega} \\ \frac{d\bar{s}}{d\omega} \end{bmatrix} = \begin{bmatrix} \bar{R} \\ 0 \end{bmatrix}$$

The determinant is

$$\bigtriangledown = (1-\omega)H'(\bar{s})[J(\bar{s}) - \bar{R}] - J'(\bar{s})(1 + H'(\bar{s})) > 0$$

and by Cramer's rule, both $\bar{R}$ and $\bar{s}$ are increasing in ω.

$$\bigtriangledown_R = \begin{vmatrix} \bar{R} & -J'(\bar{s}) \\ 0 & H'(\bar{s})(J(\bar{s}) - \bar{R}) \end{vmatrix} = \bar{R}H'(\bar{s})[J(\bar{s}) - R] > 0$$

and

$$\bigtriangledown_s = \begin{vmatrix} (1-\omega) & \bar{R} \\ -(1 + H(\bar{s})) & 0 \end{vmatrix} = \bar{R}(1 + H(\bar{s})) > 0$$

$$\frac{d\bar{R}}{d\omega} = \frac{\bigtriangledown_R}{\bigtriangledown} > 0$$
$$\frac{d\bar{s}}{d\omega} = \frac{\bigtriangledown_s}{\bigtriangledown} > 0$$

Therefore both the boundary between region 1 and 2 and the boundary between region 2 and 3 are increasing with wealth.

□

A.2 Proof of Proposition 1

Let the superscript N denote equilibrium values in the no-CBDC equilibrium. Combining the first-order condition for buyers' portfolio choice with the market-clearing condition for deposits yields

$$\frac{1}{\beta R} = u'(R(1 - k(R,\omega))\ell(R,\omega)). \tag{A.23}$$

We can write this equation as

$$(u')^{-1}\left(\frac{1}{\beta R}\right) = R(1-k)\ell. \tag{A.24}$$

The left-hand side is the demand for deposits and the right-hand side is the supply. By the assumptions of the buyers' utility function, the left-hand side is continuous and strictly increasing function of R. The demand for deposits starts at 0 when $R = 0$, increases towards q^* at $R = \frac{1}{\beta}$, and becomes vertical at $R = \frac{1}{\beta}$. The right-hand side is decreasing in R. Total loans ℓ is either

fixed at I, or continuously decreasing from I when $\ell = (\omega I)/k$. As shown in Lemma 1, the bank either continuously increases their capital ratio as R increases or keeps k fixed, which means that the deposits they supply, $(1-k)\ell$ either continuously decreases or is flat. The banks use no capital when R is low, so deposits supplied start above 0 at $R = 0$. It follows that (A.24) has a unique solution at some equilibrium interest rate $R^N(\omega)$.

I establish the possible equilibrium interest rates depending on ω.

1. If $\omega = 0$, the banks' supply of deposits is flat at I. If $\bar{R}_0 \geq \frac{1}{\beta}$, then supply will intersect with demand such that $R^N = \frac{1}{\beta}$ if $I \geq q^*$. If $I < \beta q^*$, then $R^N < \frac{1}{\beta}$.

   From the assumption on preferences, if $\bar{R}_0 < \frac{1}{\beta}$, the supply of deposits will meet the demand at some $R < \frac{1}{\beta}$ and $R^N < \frac{1}{\beta}$. I will suppose this is the case for $\bar{R}_0$ for the following cases.

2. If $\omega \geq 1$, $R^N < \frac{1}{\beta}$. The supply of deposits is flat until $\underline{R}$ and decreasing for $R > \underline{R}$. Banks are not equity-constrained and at $R = \frac{1}{\beta}$, they can choose $k = 1$. But if they are fully capital-financed, they do not issue any deposits and supply is at 0 at $R = \frac{1}{\beta}$. Therefore, deposits demand must intersect with supply at some $R^N < \frac{1}{\beta}$.

3. If $\omega_0 < \omega < 1$, deposit supply is flat for $R < \underline{R}$, decreasing for $\underline{R}(\omega) < R < \hat{R}(\omega)$, and flat for $R \geq \hat{R}(\omega)$. If $(1-\omega)I \geq \beta q^*$, the deposit supply will intersect demand at R^N where

$$R^N \begin{Bmatrix} \geq \\ < \end{Bmatrix} \frac{1}{\beta} \text{ if } (1-\omega)I \begin{Bmatrix} \geq \\ < \end{Bmatrix} \beta q^*.$$

4. If $0 < \omega \leq \omega_0$, then the three regions for deposit supply described above exist, in addition to one more region when $R > \bar{R}(\omega)$ where the supply of deposits is decreasing. At $R = \frac{1}{\beta}$, the bank must exert effort $s = s_0$, which requires $k = \omega_0$. Therefore the amount of loans the bank can make for a given ω when $(\omega/\omega_0) < 1$ is given by:

$$\ell = \frac{\omega I}{\omega_0} < I.$$

   Then the deposit market clears at R^N such that

$$R^N \begin{Bmatrix} \geq \\ < \end{Bmatrix} \frac{1}{\beta} \text{ if } (1-\omega)\frac{\omega}{\omega_0} I \begin{Bmatrix} \geq \\ < \end{Bmatrix} \beta q^*.$$

Because the bank supplies some deposits in equilibrium for all of the cases above, they will

never be fully capital financed ($k = 1$). But this means that $R(1-k) = J(s)$ will be such that $J(s) > J(s^*) = 0$ and $s < s^*$. Banks will never exert the efficient level of effort.

A.3 Proof of Proposition 2

Let $\underline{I}$ be the level of investment such that

$$1 = \beta \underline{R} u'(\underline{RI}). \tag{A.25}$$

where $\underline{R}$ is the same cutoff from the bank's solution in Lemma 1.

We proved that when $R < \underline{R}$, banks use no capital. And when $k = 0$, the bank's solution also yields s that solves $R = J(s)$ and $\ell = I$. Therefore we need to verify that if $I < \underline{I}$, $R < \underline{R}$. Equilibrium R is determined by (A.25). By taking the derivative with respect to I

$$0 = \beta R^2 u''(RI) = \{\beta u'(RI) + \beta RI u''(RI)\}\frac{dR}{dI}$$

and by our assumptions on $u(q)$ and the market clearing condition for $q = RI$:

$$\frac{dR}{dI} = \frac{-R^2 u''(q)}{u'(q) + qu''(q)} > 0.$$

we see that R is increasing with I. Therefore it would be a contradiction if $I < \underline{I}$ and $R > \underline{R}$.

Because we are focusing on cases in which banks' wealth is always at least as large as I, the banks' solution is only either in region 0 or 1. For $R > \underline{R}$, $k > 0$ and the s, k, and ℓ are determined from the banks' region 1 solution. Market clearing for deposits yields $q = R(1-k)I$ and R is determined by (42).

To verify that R is increasing with I in region 1, we rearrange (42) in terms of s and I:

$$\frac{1}{\beta} = \left[\frac{1}{\beta(H(s)+1)}\right] u'(J(s)I).$$

Taking the derivative of this with respect to I yields

$$\frac{-u''(q)J(s)}{H(s)+1} = \frac{u''(q)J'(s)[H(s)+1] - u'(q)H'(s)}{[H(s)+1]^2}\frac{ds}{dI}$$

and the left-hand side of the equation is positive while the fraction on the right-hand side is

positive, which implies that $\frac{ds}{dI}$ is positive as well. From (39),

$$-\frac{1}{\beta R^2}\frac{dR}{dI} = H'(s)\frac{ds}{dI}$$
$$\frac{dR}{dI} = -\beta R^2 H'(s) > 0$$

it must be the case that R is also increasing with I. We can also verify k is increasing with I in this region.

$$(1-k)\frac{dR}{dI} - J'(s)\frac{ds}{dI} = R\frac{dk}{dI} \tag{A.26}$$

The left-hand side of the equation is positive, so for any positive R, this also implies $\frac{dk}{dI} > 0$.

A.4 Proof of Proposition 3

Defining the bounds:

For a given ω, we have the banks' cutoffs $\hat{R}(\omega)$ and $\bar{R}(\omega)$, let $\hat{I}(\omega)$ be the level of investment such that

$$1 = \beta\hat{R}(\omega)u'[\hat{R}(\omega)(1-\omega)\hat{I}(\omega)] \tag{A.27}$$

and let $\bar{I}(\omega)$ be the level of I such that

$$1 = \beta\bar{R}(\omega)u'[\bar{R}(\omega)(1-\omega)\bar{I}(\omega)], \tag{A.28}$$

for a given ω.

Verifying equilibrium solutions: Next, we verify the four possible equilibrium cases.

1. Region 0 exists for $I < \underline{I}$, regardless of ω, and so its existence is verified in the proof of Propositon 2.

2. Region 1 as defined in Proposition 2 also exists, but the upper bound of that region is now $\hat{R}(\omega)$. We proved earlier that $\frac{dk}{dI}$ is positive in this region. It would be a contradiction if $I < \hat{I}(\omega)$ but $k > \omega$, so $k < \omega$ in this region.

3. In region 2, I is high enough for a given ω that the banks' solution is in region 2, and $k = \omega$, $\ell = I$, and s is chosen to satisfy $J(s) = R(1-\omega)$. To verify that if $I \geq \hat{I}(\omega)$, $R > \hat{R}(\omega)$, we see that R is increasing with I when k is fixed:

$$\frac{dR}{dI} = \frac{-R^2(1-\omega)u''(q)}{u'(q) + qu''(q)} > 0.$$

If $\omega \geq \omega_0$, the upper bound of R in this equilibrium region is $R = \frac{1}{\beta}$. If $\omega < \omega_0$, then the upper bound is $\bar{I}(\omega)$.

4. If the equilibrium interest rate is greater than $\bar{R}(\omega)$ for a given $\omega < \omega_0$, then we know that the banks' solution yields s, k, and ℓ that satisfy (49)-(51). Therefore, we need to verify that if $I > \bar{I}(\omega)$, $R > \bar{R}(\omega)$.

   From (49), the interest rate can be written in terms of s:

$$R(s) = \frac{sy - c(s) + H(s)J(s)}{1 + H(s)} \tag{A.29}$$

   and we can also denote the market-clearing q in terms of s and I, using (50) and (51):

$$k(s) = 1 - \frac{J(s)}{R(s)} \tag{A.30}$$

$$\ell(s) = \frac{\omega I}{k(s)} \tag{A.31}$$

$$q(s) = R(s)[1 - k(s)]\ell(s) = \frac{J(s)\omega I}{k(s)}. \tag{A.32}$$

   The equilibrium interest rate is then determined by

$$\frac{1}{\beta} = R(s)u'(q(s)).$$

   This deposit interest rate is increasing in s:

$$R'(s) = \frac{H'(s)J(s)[1 + H(s)] + H'(s)[sy - c(s) + H(s)J(s)]}{[1 + H(s)]^2} > 0,$$

   since the numerator can be simplified as

$$\begin{aligned} H'(s)[J(s) - sy + c(s)] &= H'(s)[-sc'(s) + c(s)] \\ &= H'(s)[-(sc'(s) - c(s))] \\ &> 0. \end{aligned}$$

The deposit supply is decreasing in s:

$$\begin{aligned} q'(s) &= \frac{[J'(s)wIR(s) + J(s)\omega IR'(s)](R(s) - J(s)) - (R'(s) - J'(s))J(s)\omega IR(s)}{(R(s) - J(s))^2} \\ &= \frac{\omega I}{(R(s) - J(s))^2}\Big\{[J'(s)R(s) + J(s)R'(s)](R(s) - J(s)) - (R'(s) - J'(s))J(s)R(s)\Big\} \\ &= \frac{\omega I}{(R(s) - J(s))^2}\Big\{J'(s)R(s)^2 - J(s)^2R'(s)\Big\} \\ &< 0. \end{aligned}$$

Then we can see that effort is increasing with I:

$$\begin{aligned} 0 &= \Big\{R'(s)u'(q(s)) + R(s)q'(s)u''(q(s))\Big\}\frac{ds}{dI} + R(s)q(s)u''(q(s)) \\ \frac{ds}{dI} &= \frac{-R(s)q(s)u''(q(s))}{R'(s)u'(q(s)) + R(s)q'(s)u''(q(s))} > 0. \end{aligned}$$

Finally, the derivative of the zero-profit condition implies that R is increasing with I in this region:

$$H'(s)\Big[J(s) - R\Big]\frac{ds}{dI} = [1 + H(s)]\frac{dR}{dI}.$$

Characteristics of investment boundaries: We establish that the investment boundaries between equilibrium regions are increasing in ω. Let $\hat{q}(\omega) = \hat{R}(\omega)(1-\omega)\hat{I}(\omega)$. For ease of notation, I will drop the ω function notation for the following analysis. Taking the derivative of (47) yields

$$0 = \Big\{u'(\hat{q}) + \hat{q}u''(\hat{q})\Big\}\frac{d\hat{R}}{d\omega} + \hat{R}^2(1-\omega)u''(\hat{q})\frac{d\hat{I}}{d\omega}.$$

Since we proved that $\frac{d\hat{R}}{d\omega} > 0$ in Lemma 1, this implies that $\hat{I}$ is increasing with ω:

$$\frac{-\{u'(\hat{q}) + \hat{q}u''(\hat{q})\}\frac{d\hat{R}}{d\omega}}{\hat{R}^2(1-\omega)u''(\hat{q})} = \frac{d\hat{I}}{d\omega} > 0.$$

Similarly, taking the derivative of

$$1 = \beta\bar{R}u'(\bar{R}(1-\omega)\bar{I})$$

with respect to ω and using $\frac{d\hat{R}}{d\omega} > 0$ will also show the boundary between equilibrium region 2 and 3 is increasing with ω.

Last, we can verify that $\underline{R} < \hat{I}(\omega) < \bar{I}(\omega)$. Suppose to the contrary that $\underline{I} \geq \hat{I}$. By

definition, $\underline{I}$ is the level of investment such that the equilibrium interest rate is $\underline{R}$. It follows that effort at $\underline{I}$ is $\underline{s}$. By similar logic, effort at $\hat{I}$ is $\hat{s}$. We proved earlier that $\underline{s} < \hat{s}$, which implies $J(\underline{s}) > J(\hat{s})$. The amount of deposits supplied for the equilibrium R satisfies

$$\underline{q} = J(\underline{s})\underline{I} > J(\hat{s})(1-\omega)I = \hat{q}.$$

Properties of $u(q)$ are such that $u'(\underline{q}) < u'(\hat{q})$, which implies that $\underline{R} > \hat{R}$, from (A.25) and (A.27). But this would mean

$$H(\underline{s}) = \frac{1}{\beta \underline{R}} - 1 < \frac{1}{\beta \hat{R}} - 1 = H(\hat{s}), \tag{A.33}$$

which implies $\underline{s} > \hat{s}$, which is a contradiction. Therefore, it must be that $\underline{I} < \hat{I}$.

We know that $\hat{R}(\omega) < \bar{R}(\omega)$ for a given ω, and we also know that in equilibrium region 2, $\frac{dR}{dI} > 0$. Therefore it would be a contradiction if $\hat{R} < \bar{R}$ and $\hat{I} \geq \bar{I}$. This proves that $\hat{I}(\omega) < \bar{I}(\omega)$.

A.5 Proof of Proposition 4

When R_e is set so that $R_e > R^N$, the equilibrium interest rate rises to R_e. The household demand for liquid assets is determined by

$$\frac{1}{\beta R_e} = u'(R_e(e^E + \delta^E))$$

and the quantity of CBDC held specifically is

$$e^E = \frac{1}{R_e}(u')^{-1}\left(\frac{1}{\beta R_e}\right) - \delta^E. \tag{A.34}$$

Buyers' demand for liquid assets increases with the interest rate, so the first-term on the left-hand side of (A.34) is increasing in the R. The amount of deposits banks will supply depends on which equilibrium region R_e is in, which determines k and ℓ. As k and ℓ in the banks' solution are nonincreasing in R, total deposit supply will also be nonincreasing for $R_e > R^N$ and $\delta^E \leq \delta^N$. Therefore buyers hold a strictly positive amount of CBDC when $R_e > R^N$.

A.6 Proof of Lemma 2

Total welfare is given by

$$\mathcal{W} = [-1 + \beta(sy - c(s))]\ell + \beta[u(q) - q]$$

where s, ℓ, and q depend on R. Taking the derivative of $\mathcal{W}$ with respect to R yields

$$\frac{d\mathcal{W}}{dR} = \Big\{\beta[y - c'(s)]I\Big\}\frac{ds}{dR} + \beta[u'(q) - 1]\frac{dq}{dR}. \tag{A.35}$$

Since total q no longer depends solely on the banks' deposit supply, q directly increases with R:

$$\frac{dq}{dR} = -\frac{u'(q)}{Ru''(q)} < 0. \tag{A.36}$$

The effects of higher R on bank effort follow from the results of Lemma 1. Next, I analyze the effect on total welfare depending on each equilibrium region.

Equilibrium region 0: In this reason, we have established that $k = 0$, $\ell = I$, and s is determined by $J(s) = R$ and that there is a negative effect on effort when R increases:

$$\frac{ds}{dR} = \frac{ds}{dR} = \frac{1}{J'(s)} < 0.$$

Substituting (A.36) and (A.6) into (A.35) yields:

$$\frac{1}{\beta}\frac{d\mathcal{W}}{dR} = [y - c'(s)]I\left[\frac{1}{J'(s)}\right] + (u'(q) - 1)\left[\frac{-u'(q)}{Ru''(q)}\right]. \tag{A.37}$$

The first term can be simplified as

$$\frac{y - c'(s)}{y - c'(s) - sc''(s)}I = -H(s)I.$$

Using the fact that $u'(q) = \frac{1}{\beta R}$ around equilibrium and that in this region

$$u'(q) - 1 = \frac{1}{\beta R} - 1 > H(s)$$

the effect of raising R in the neighborhood of the CBDC equilibrium is:

$$\begin{aligned}\frac{1}{\beta}\frac{d\mathcal{W}}{dR} &= -H(s)I + (u'(q) - 1) \\ &> -H(s)I + H(s)\frac{-u'(q)}{Ru''(q)} \\ &= H(s)\left[\frac{-u'(q)}{Ru''(q)} - I\right] \\ &> 0.\end{aligned}$$

This follows from the assumptions on $u(q)$,

$$-qu''(q)/u'(q) < 1 \longrightarrow q < \frac{-u'(q)}{u''(q)}$$

and the fact that q is now greater than or equal to the deposit supply, $q \geq R(1-k)I$:

$$\frac{-u'(q)}{Ru''(q)} > \frac{q}{R} \geq I.$$

Equilibrium region 1: In this region, s is determined by $(1/\beta R) - 1 = H(s)$ and is increasing in R:

$$\frac{ds}{dR} = -\frac{1}{\beta R^2 H'(s)} > 0. \tag{A.38}$$

Since q is also increasing with R, then total welfare is increasing, according to (A.35).

Equilibrium region 2: In this region, $k = \omega$ and s is determined by $J(s) = R(1-\omega)$. Effort decreases with R:

$$\frac{ds}{dR} = \frac{1-\omega}{J'(s)} < 0. \tag{A.39}$$

The change in total welfare with respect to R is

$$\frac{1}{\beta}\frac{d\mathcal{W}}{dR} = -H'(s)(1-\omega)I + (u'(q) - 1)\frac{-u'(q)}{Ru''(q)}. \tag{A.40}$$

The sign of this expression is ambiguous. While $H(s) > \frac{1}{\beta R} - 1 = (u'(q) - 1)$ in this region,

$$\frac{-u'(q)}{Ru''(q)} > \frac{q}{R} \geq (1-\omega)I$$

so one effect does not always dominate the other.

Note that raising R around $R = \frac{1}{\beta}$ has a negative effect on welfare in region 2. At $R = \frac{1}{\beta}$, households will be able to afford q^* and $u'(q^*) = 1$, so the second term in (A.35) goes to zero. Because $s < s^*$ and $J'(s) < 0$, we see that

$$\frac{1}{\beta}\frac{d\mathcal{W}}{dR} = [y - c'(s)]\frac{1-\omega}{J'(s)}I < 0.$$

The lower bound of the equity-constrained regions occurs at $\hat{I}(\omega)$, $R = \hat{R}(\omega)$ and $s = \hat{s}$. We also know that

$$H(\hat{s}) = \frac{1}{\beta\hat{R}} - 1$$

Therefore

$$\begin{aligned}
\frac{1}{\beta}\frac{d\mathcal{W}}{dR} &= -H(\hat{s})(1-\omega)\hat{I} + \left(\frac{1}{\beta\hat{R}} - 1\right)\left(\frac{-u'(q)}{Ru''(q)}\right) \\
&= H(\hat{s})\left[\left(\frac{-u'(q)}{Ru''(q)} - (1-w)I\right)\right] \\
&> H(\hat{s})\left[\left(\frac{q}{R} - (1-w)I\right)\right] \\
&\geq 0
\end{aligned}$$

The marginal change in total welfare is a strictly decreasing function, under some further assumptions on the utility and cost function. I will assume that the fourth derivative of $c(s)$ is zero and that the third derivative of $u(q)$ is positive.

$$\begin{aligned}
\frac{1}{\beta}\frac{d^2\mathcal{W}}{dR^2} = &\left\{\frac{[c'''(s)J'(s) - (y-c'(s))3c'''(s)](J'(s))^2 - [c''(s)J'(s) - (y-c'(s))J''(s)](2J''(s))}{(J'(s))^4}\right\}\frac{ds}{dR} \\
&+ \left\{\frac{-u'(q)}{R} + (u'(q)-1)\left[\frac{-u''(q)^2R - Ru'(q)u'''(q)}{(Ru''(q))^2}\right]\right\}\frac{dq}{dR} \\
&+ \frac{y-c'(s)}{\beta H'(s)R^3}I - (u'(q)-1)\left[\frac{-u'(q)}{R^2u''(q)}\right]
\end{aligned}$$

The part of the first term in brackets is overall positive, so with $ds/dR < 0$ in region 2, the first term is negative. The part of the second term in brackets is overall negative, and with $dq/dR > 0$ the second term is also negative. The last two terms are also negative. Since this is a continuous function that is strictly decreasing, there exists an $\tilde{I}(\omega) \in (\hat{I}(\omega), \frac{\beta q^*}{1-\omega})$ such that the equilibrium interest rate and allocations satisfy $d\mathcal{W}/dR = 0$.

Equilibrium region 3: In this region, s is determined by the zero-profit condition (A.22). An increase in R increases s and k, which will also decrease ℓ. To consider the change in

total welfare, we must also consider the change in loans when R changes. Thus we analyze the following equation:

$$\frac{d\mathcal{W}}{dR} = \Big\{ -1 + \beta[sy - c(s)]\Big\}\frac{d\ell}{dR} + \beta(y - c'(s))\ell\frac{ds}{dR} + \beta(u'(q) - 1)\frac{dq}{dR}. \tag{A.41}$$

In this region, derivatives for s, k, and ℓ are:

$$\frac{ds}{dR} = \frac{1 + H(s)}{H'(s)[J(s) - R]} > 0 \tag{A.42}$$

$$\frac{dk}{dR} = \frac{1}{R}\left[(1 - k) - J'(s)\frac{ds}{dR}\right] > 0 \tag{A.43}$$

$$\frac{d\ell}{dR} = -\frac{\ell}{k}\frac{dk}{dR} < 0. \tag{A.44}$$

The second and third terms in (A.41) are positive. Let

$$-1 + \beta(sy - c(s)) \tag{A.45}$$

be the banks' profits when fully capital financed. If $-1+\beta(s^\dagger y - c(s^\dagger)) < 0$, it is possible that the first term is positive. By Assumption 2, $-1 + \beta(s^* y - c(s^*)) > 0$ which means there exists some $\grave{s}$ such that $-1 + \beta(\grave{s}y - c(\grave{s})) = 0$. And because (A.45) is monotonically increasing in s, this $\grave{s}$ is unique. If the benchmark equilibrium $s(R) < \grave{s}$, total welfare will increase as R increases. If $s(R) > \grave{s}$, then the first term of (A.41) is negative. Otherwise, if $-1 + \beta(s^\dagger y - c(s^)) \geq 0$, the first term will be negative for the relevant values of $s \in (s^\dagger, s^*)$.

I check whether it is optimal to raise R to $\frac{1}{\beta}$. I substitute (A.43) and (A.44) into (A.41) and rearrange it as:

$$\frac{d\mathcal{W}}{dR} = \left\{\frac{J'(s)}{k}[-1 + \beta(sy - c(s))] + (y - c'(s))\right\}\beta\ell\frac{ds}{dR} - \frac{(1 - k)\ell}{Rk}[-1 + \beta(sy - c(s))] + \beta[u'(q) - 1]\frac{dq}{dR} \tag{A.46}$$

In the neighborhood of $R = \frac{1}{\beta}$, $u'(q^*) = 1$ and $q \approx q^*$, so the last term of (A.45) decreases to zero. To satisfy (A.22) at $R = \frac{1}{\beta}$, the following will hold

$$-1 + \beta[sy - c(s)] = H(s)k.$$

Then the first-term of (A.41) is also equal to zero in the neighborhood of $R = \frac{1}{\beta}$:

$$\begin{aligned}\frac{J'(s)}{k}[-1 + \beta(sy - c(s))] + (y - c'(s)) &= -H(s)J'(s) + (y - c'(s)) \\ &= -(y - c'(s)) + (y - c'(s)) \\ &= 0.\end{aligned}$$

Then we are left to assess the sign of the second term in (A.46),

$$\frac{(1-k)\ell}{Rk}[-1 + \beta(sy - c(s))]$$

At $R = \frac{1}{\beta}$ in region 3, $s(R) = s_0$. To prove that $s_0 < s^*$ and $k < 1$, suppose instead that $s_0 = s^*$. Then we have

$$\begin{aligned}s_0 y - c(s_0) - \frac{1}{\beta} + H(s_0)[J(s_0) - \frac{1}{\beta}] = s^* y - c(s^*) - \frac{1}{\beta} \\ > 0\end{aligned}$$

However this means that $\lambda_3 > 0$ in the banks' problem and $\ell = I$. When $s = s^*$, the banks' incentive constraint $R(1-k) = J(s^*) = 0$ also implies that $k = 1$. However both cannot be true because this means that $\omega I = I$ ($W = I$), when total wealth W is strictly less than I in region 3. Therefore

$$\frac{d\mathcal{W}}{dR} < 0$$

in the neighborhood of $R = \frac{1}{\beta}$.

We proved earlier that $d\mathcal{W}/dR > 0$ at the lower bound $\hat{R}(\omega)$ of the wealth-constrained regions. We have just shown that (A.41) is also negative at $R = \frac{1}{\beta}$ ($d\mathcal{W}/dR > 0$). So if the upper bound of wealth-constrained regions is in equilibrium region 3, we also know that there exists some level(s) of I such that $d\mathcal{W}/dR = 0$ by the intermediate value theorem.

B Additonal details

B.1 First-best bank problem

Suppose that buyers could contract directly on effort. The bank's problem is

$$\max_{s,k,\ell,Z} -k\ell + \beta[s(y\ell - Z) - c(s)\ell] \tag{B.1}$$

subject to

$$sZ \geq R(1-k)\ell \qquad (\lambda_1) \tag{B.2}$$
$$W \geq k\ell \qquad (\lambda_2) \tag{B.3}$$
$$\ell \leq I \qquad (\lambda_3) \tag{B.4}$$
$$k \geq 0 \qquad (\mu_1) \tag{B.5}$$
$$k \leq 1 \qquad (\mu_2) \tag{B.6}$$
$$\ell \geq 0. \qquad (\mu_3) \tag{B.7}$$

First-order conditions:

$$\beta[y\ell - Z - c'(s)\ell] + \lambda_1 Z = 0 \tag{B.8}$$
$$-\ell - \lambda_2\ell + \mu_1 - \mu_2 = 0 \tag{B.9}$$
$$-k + \beta[sy - c(s)] - \lambda_1 R(1-k) + \lambda_3 - \mu_3 = 0 \tag{B.10}$$
$$-\beta s + \lambda_1 s = 0 \tag{B.11}$$

The last first-order condition simplifies to $\lambda_1 = \beta$. Therefore $sZ \geq R(1-k)\ell$ will always hold with equality.